An Overview of Monads and Comonads
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Outline

I. The λ-Calculus

The simply typed λ-calculus consists of:

- **Types**: $X, Y, \dots, A \times B,\ A \to B, \dots$
- **Terms**: $x : A,\ b : B,\ \langle a, b \rangle,\ \lambda x. b(x), \dots$
- (**Equations** $s = t : A$)

Usually presented as a deductive system by rules of inference like:

$$\frac{a : A \qquad b : B}{\langle a, b \rangle : A \times B} \qquad \frac{c : A \times B}{fst(c) : A \qquad (snd(c) : B)}$$

$$\frac{a : A \qquad f : A \to B}{f(a) : B} \qquad \frac{x : A \vdash b(x) : B}{\lambda x \!:\! A.\, b(x) : A \to B}$$

Curry-Howard

The system is usually regarded as a calculus for specifying and evaluating functions, but it also has another interpretation:

- The types are "propositions" and their terms are "proofs", which are being derived.

This is known as the **Curry-Howard correspondence**:

0	1	$A + B$	$A \times B$	$A \to B$	$\sum_{x:A} B(x)$	$\prod_{x:A} B(x)$
$\bot$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\exists_{x:A} B(x)$	$\forall_{x:A} B(x)$

It even extends to predicate logic, and beyond, but we will just be concerned with the propositional fragment today.

Curry-Howard

- These are indeed very different ideas: the “proof-relevant” interpretation is modelled by sets and functions, while the “proof-irrelevant” one is modelled by subsets of a fixed set.
- Under the former, there may be many different terms $a : A$ of a given type; but under the latter interpretation, one typically just considers whether a proposition is *provable* – i.e. has a proof $a : A$.

Curry-Howard

Of course, different proofs of a given proposition may correspond to different terms:

$$\dfrac{\dfrac{\dfrac{\dfrac{x:A, y:B \vdash x : A \qquad x:A, y:B \vdash y : B}{x:A, y:B \vdash \langle x, y\rangle : A \times B}}{x:A, y:B \vdash p_1\langle x, y\rangle : A}}{x:A \vdash \lambda y:B.\, p_1\langle x, y\rangle : B \to A}}{\vdash {\color{red}\lambda x:A\lambda y:B.\, p_1\langle x, y\rangle} : A \to (B \to A)}$$

$$\dfrac{\dfrac{x:A, y:B \vdash x : A}{x:A \vdash \lambda y:B.\, x : B \to A}}{\vdash {\color{red}\lambda x:A\lambda y:B.\, x} : A \to (B \to A)}$$

The Curry-Howard correspondence also gives rise to a connection between *functions* and proofs, called the BHK interpretation.

II. Cartesian closed categories

The Curry-Howard correspondence can be understood in terms of category theory as the recognition that the things being related – namely, proofs in propositional logic and terms in the lambda calculus, share a common structure, namely that of being a *cartesian closed category*.

Definition

A category $\mathbb{C}$ is *cartesian closed* if it has the following structure:

- a terminal object 1.
- for every pair of objects A, B a product $A \times B$, with projection maps $A \leftarrow A \times B \rightarrow A$,
- for every pair of objects A, B an exponential B^A with evaluation map $B^A \times A \longrightarrow B$.

This structure is also had by functions between sets (and similar), which also helps to "explain" the BHK interpretation.

II. Cartesian closed categories

Examples of CCCs include:

- Sets and functions.
- Boolean algebras, such a powersets $\mathcal{P}(A)$, and Lindenbaum-Tarski algebras of propositional logics.
- More general posets, like the collection of open sets $\mathcal{O}(X)$ of a topological space X.
- Families of sets $(A_i)_{i \in I}$ indexed over some set I, or even a poset or category I ("Kripke models").

The λ-calculus can be shown to be deductively complete with respect to such semantics, just like the propositional calculus.

III. Modalities

Now we want to consider adding another ingredient to propositional logic, namely modal operators like $\Box P$ and $\Diamond P$.

In propositional logic, such operators can of course represent many different things:

- necessity, possibility
- knowledge, belief
- always, sometimes,
- should, may,
- etc.

III. Modalities

From the many different interpretations, we see that – unlike the conjunction $P \wedge Q$ and implication $P \Rightarrow Q$ – the meaning of such modal operators $\Box P$ is not "fixed". In particular, the choice of rules will be determined by the intended interpretation or range of interpretations.

For example, whether we have

$$\diamond P \wedge \diamond Q \to \diamond(P \wedge Q)$$

depends on the interpretation of $\diamond$: if it's belief, then this might be reasonable, but if it's possibility, then it's less so.

III. Modal type theory

We want to extend the Curry-Howard correspondence to include also such modalities:

$\bot$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\Box A$	$\Diamond A$
0	1	$A + B$	$A \times B$	$A \to B$	$?A$	$?A$

III. Modal type theory

- In logic, the modality $\diamond P$ on propositions usually satifies laws such as

$$P \Rightarrow \diamond P, \qquad \diamond\diamond P \Rightarrow \diamond P, \qquad (P \Rightarrow Q) \Rightarrow (\diamond P \Rightarrow \diamond Q)$$

- Under Curry-Howard, such a modality is encoded in **simple type theory** by rules such as:

$$\frac{\Gamma \vdash A \ \texttt{type}}{\Gamma \vdash \diamond A \ \texttt{type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash a^{\diamond} : \diamond A}$$

  and so on.

- Finding the right type theory is guided semantically by the concept of a **monad** $T : \mathbb{C} \longrightarrow \mathbb{C}$ on a category $\mathbb{C}$, with may be thought of as the category of types and terms.
- The basic idea goes back to the ground-breaking paper of E. Moggi, Computational lambda-calculus and monads, 1988.

III. Modal type theory

- Another common logical modality $\Box p$ satifies the **dual laws**,

$$\Box P \Rightarrow P, \qquad \Box P \Rightarrow \Box\Box P, \qquad (P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q)$$

- Encoding such a **comonadic modality** in type theory is more difficult – especially in dependent type theory. This is the focus of current research by Zwanziger and others, drawing on earlier work of Pfenning, Biermann and de Paiva, and others.
- Such a modality is modelled semantically by a **comonad** $S : \mathbb{C} \longrightarrow \mathbb{C}$ on the category of types and terms.

IV. Monads and Comonads

Definition

A **monad** (T, η, μ) on a category $\mathbb{C}$ consists of

- a functor $T : \mathbb{C} \longrightarrow \mathbb{C}$,
- a natural transformation $\eta : 1_{\mathbb{C}} \longrightarrow T$,
- a natural transformation $\mu : T^2 \longrightarrow T$,

satisfying equations stating that μ is associative and η is a unit:

$$\mu_A \circ \mu_{TA} = \mu_a \circ T\mu_A, \qquad \mu_A \circ \eta_{TA} = 1_{TA} = \mu_A \circ T\eta_A \,.$$

IV. Monads and Comonads

Examples of monads include:

- possibility $\diamond P$,
- powerset $\mathcal{P}(A)$,
- the list monad $X^* = \sum_{n \in \mathbb{N}} X^n$,
- partiality $X + 1$,

IV. Monads and Comonads

Definition

A **comonad** (S, ϵ, δ) on a category $\mathbb{C}$ consists of

- a functor $S : \mathbb{C} \longrightarrow \mathbb{C}$,
- a natural transformation $\epsilon : S \longrightarrow 1_{\mathbb{C}}$,
- a natural transformation $\delta : S \longrightarrow S^2$,

satisfying the monad equations in $\mathbb{C}^{op}$.

IV. Monads and Comonads

Examples of comonads include:

- necessity $\Box P$,
- the discrete space of points $|X|$ of a space X,
- the discrete category of objects $\mathbb{C}_0$ of a category $\mathbb{C}$,
- infinite streams $(x_0, x_1, \ldots) \in X^\omega$ of elements of a set X.

IV. (Co)Monads as Kleisli (Co)Triples

There is an alternate presentation of the notion of a (co)monad, which is often more convenient for the purposes of type theory.

Definition

A **Kleisli triple** $(T, \eta, ^*)$ on a category $\mathbb{C}$ consists of

- an operation $C \mapsto TC$ on objects,
- a family of arrows $\eta_C : C \to TC$,
- a operation taking each $f : A \to TB$ to an $f^* : TA \to TB$,

satisfying the equations:

$$(\eta_A)^* = 1_{TA}, \qquad f^* \circ \eta_A = f, \qquad g^* \circ f^* = (g^* \circ f)^* .$$

This specification can be shown to be equivalent to the previous one.

The dual notion of **Kleisli cotriple** is as expected.