Spatial Models of Higher-Order S4

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Abstract Motivation

How do we complete the following table?

Level	Geometric Model	$S4$ Modal Logic	Logic of Verifiability
1	Topological Spaces	Propositional	Propositional
2	???	Predicate	Predicate

More Concrete Motivation

A notion of space designed to have mathematical structures (structured sets) as the "points"

1. Built-in respect for isomorphism
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More Concrete Motivation

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Awodey and Kishida (2008) and Garner (2010) are potential answers to our question, with property (2) but not (1).

Outline

Types

- There are basic types E, T.
- If A and B are types, then so is $A \to B$.
- The type $\mathrm{E} \to \mathrm{T}$ for predicates on entities, $(\mathrm{E} \to \mathrm{T}) \to \mathrm{T}$ for predicates on predicates on entities, etc.

Term Calculus

(Non-Modal) Logic

- $\dfrac{}{\cdot \mid \top : \mathsf{T}}$, and similarly for $\bot$
- $\dfrac{\phi : \mathsf{T} \qquad \psi : \mathsf{T}}{\phi \wedge \psi : \mathsf{T}}$, and similarly for $\vee$ and $\Rightarrow$.
- $\dfrac{\phi : \mathsf{T}}{\neg \phi : \mathsf{T}}$
- $\dfrac{\Gamma, x : A \mid \phi : \mathsf{T}}{\Gamma \mid \forall x . \phi : \mathsf{T}}$, and similarly for $\exists$.
- $\dfrac{t : A \qquad u : A}{t =_A u : \mathsf{T}}$

Term Calculus

Variables and Function Types

- $$\frac{}{\Gamma, x : A, \Delta \mid x : A}$$
- $$\frac{\Gamma, x : A \mid t : B}{\Gamma \mid \lambda x.t : A \to B}$$
- $$\frac{t : A \to B \qquad u : A}{tu : B}$$

Note that we will have: $\lambda x.(tx) \equiv t \qquad (\lambda x.t)x' \equiv t[x'/x]$

Term Calculus

Modality

($\Box$ Form.).

$$\frac{\Gamma \mid s_1 : \mathsf{E}, ..., \Gamma \mid s_n : \mathsf{E} \qquad x_1 : \mathsf{E}, ..., x_n : \mathsf{E} \mid \phi(x_1, ..., x_n) : \mathsf{T}}{\Gamma \mid \Box\phi([s_1], ..., [s_n]) : \mathsf{T}}$$

Entailment

Logic

- We axiomatize entailment as a judgement of form

$$\Gamma \mid \phi \vdash \psi$$

- Rules for (non-modal) logic are as usual
- In particular, existential generalization and universal instantiation are valid.

Entailment

Modality

(ML1). $$\frac{x : \mathsf{E} \mid \phi(x) \vdash \psi(x)}{x : \mathsf{E} \mid \Box\phi([x]) \vdash \Box\psi([x])}$$

(ML2). $x : \mathsf{E} \mid \Box\phi([x]) \vdash \phi(x)$

(ML3). $x : \mathsf{E} \mid \Box\phi([x]) \vdash \Box\Box\phi([[x]])$

(ML4). $x : \mathsf{E} \mid \Box\phi([x]) \wedge \Box\psi([x]) \vdash \Box(\phi \wedge \psi)([x])$

(ML5). $x : \mathsf{E} \mid \top \vdash \Box\top$

Entailment

Modality (continued)

In particular the following principles are derivable:

(Necessitation). $$\frac{x : \mathsf{E} \mid \top \vdash \phi(x)}{x : \mathsf{E} \mid \top \vdash \Box\phi([x])}$$

(K). $$x : \mathsf{E} \mid \Box(\phi \Rightarrow \psi)([x]) \vdash \Box\phi([x]) \Rightarrow \Box\psi([x])$$

(Converse Barcan Principle). $$\frac{x : \mathsf{E}, y : \mathsf{E} \mid \phi(x, y) : \mathsf{T}}{x : \mathsf{E} \mid \Box(\forall y.\phi)([x]) \vdash \forall y.\Box\phi([x, y])}$$

Sheaf Semantics

Awodey and Kishida (2008) introduce a topological model in which higher-order $S4$ can be interpreted.

This model is like a (variable-domain) Kripke model in that it has a set of entities at each point of the topological space, and open formulas are interpreted as subsets (of the set of entities) at each point.

Entities

We will think of our entities as living together in a space, with a projection down to the index set.

$$F_p = \pi^{-1}(\{p\}).$$

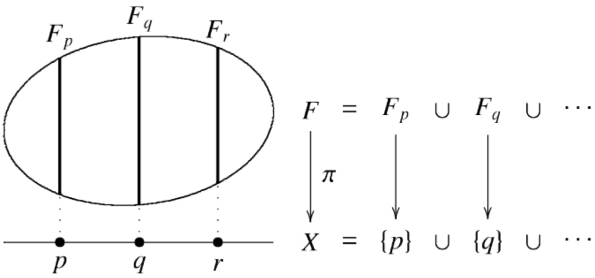

Figure: Source: Awodey-Kishida

Formulas

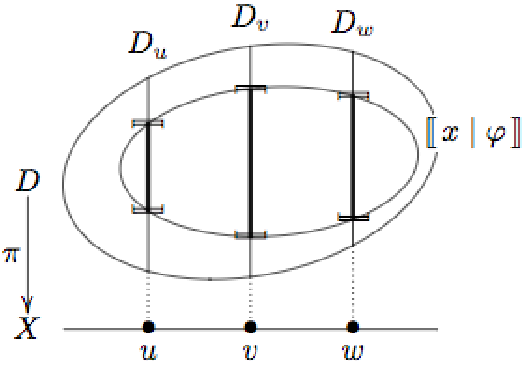

Figure: Interpretation of an open formula ϕ (Source: Awodey-Kishida)

Sheaf Semantics

For this to work, our $\pi : F \to X$ must be a *sheaf*.

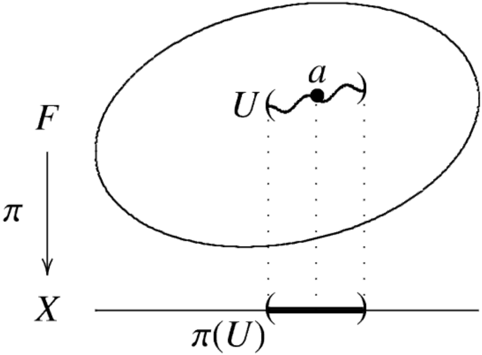

Figure: A sheaf (Source: Awodey-Kishida)

A Sheaf

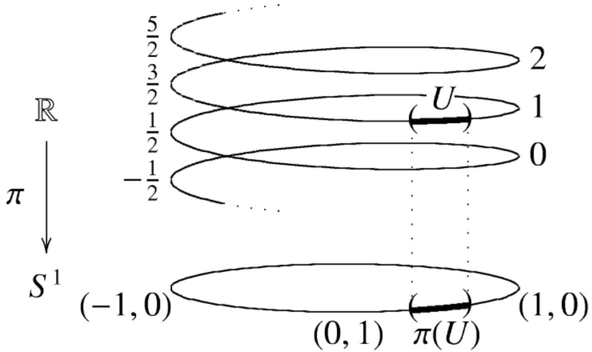

Figure: Example of a sheaf (Source: Awodey-Kishida)

Returning to "Ontological Information"

A sheaf is a generalization of an open set!

Proposition

A continuous function $f : U \to X$ is (up to homeomorphism) an inclusion of an open set of X iff f is a sheaf and, for all $x \in X$, $f^{-1}(x) \cong \emptyset$ or $f^{-1}(x) \cong \{\emptyset\}$.

2-Topology

Remember, we are seeking to generalize the "generalized interior operation" and "generalized open sets" of the Awodey-Kishida model. Such thinking can lead us to the following definition:

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  - T preserves finite limits

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Our type T will be interpreted as the functor $\mathcal{T} : G \to Set$ defined by $T(g) = Bool$ for all $g \in G$.

Interpretation (continued)

The (non-modal) logic is interpreted as in a Kripke model, except the interpretations are as in a Kripke model, except the interpretations respect all isomorphisms (including automorphisms!).

The formula $\Box\phi$ is interpreted as interior of the underlying (1-)topological space.

Examples

- (Topological) Sheaf Semantics
- Groups (of bounded cardinality)
- Rings (of bounded cardinality)
- In fact, the groupoid of structures X (of bounded cardinality), whenever X is a mathematical structure whose axioms can be written down in a FOL of verifiability with a finite signature

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Garner suggests we think of these last as a canonical "generalized space of [X's] equipped with the Scott [2-]topology".

What can we do with 2-Topology?

- Verifiability model (??)
- Bridge to topos theory (2-topology is intermediate between topology and topos theory)
- "Higher" analogs of topological notions (separation axioms, perhaps 2-(topological) domain theory, Stone duality for predicate logic, ...)

Thanks!