Montague's *Intensional Logic* as Comonadic Type Theory

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Any errors or infelicities are, of course, my own.

Contents

Chapter 1

Introduction

The compositional, denotational approach to natural language semantics pioneered by Richard Montague (1970a, 1970b, 1973) was underwritten by his development of an extension of higher-order modal logic suitable for interpreting a wide range of natural language sentences. A precise formulation of syntax and semantics for what he named 'intensional logic' is provided in *The Proper Treatment of Quantification in Ordinary English* (Montague 1973).

Potential to apply category theory to the study of Montague's intensional logic has been noted in the literature. Lambek (1988) observed some resemblance between the semantics Montague offers for his logic and the topos $\mathbf{Set}^W$, for a set W of 'possible worlds'. A category-theoretic approach to the semantics of predicate S4 modal logic took shape around this time, with the general formulation coming in Reyes and Zolfaghari (1991). Reyes (1991) made use of this approach in presenting a linguistically motivated type theory and furthermore mooted a connection to Montague's logic. Awodey, Buchholtz, and Zwanziger (2016) provided the precise categorical semantics. The present thesis expands on that work, introducing a simplified syntax, a full-fledged deductive calculus, and a more user-friendly presentation of the semantics which takes the notion of comonad as basic.

In Chapter 2, we introduce a revised version of Montague's logic, **MIL**, different from the original most notably in that the *de re*/*de dicto* distinction is maintained by the noncommutativity of substitution with the necessity and ‘intension’ operators, rather than by abandoning the principle of β-reduction. As a result, **MIL** is the first system to allow β-reduction into oblique contexts. **MIL** also provides a simplified approach to the syntax of modal type theories such as those considered in Bierman and de Paiva (1999) and Pfenning and Davies (2000).

Chapter 3 gives a categorical semantics of **MIL**. The approach interprets Montague’s ‘intension’ and ‘extension’ operators as operations of derived from the structure of a comonad. Finally, in Chapter 4, we recover the interpretation from Montague (1973) as a special case of the categorical semantics and indicate how to subsume other examples such as the Boolean-valued semantics of Gallin (1975).

Chapter 2

Montague's Intensional Logic

The system **MIL** is a revised version of Montague's 'intensional logic'. The central insight of this approach, originating in Awodey, Buchholtz, and Zwanziger (2015), is the identification of Montague's $\hat{}$ ('intension') and $\check{}$ ('extension') operators as term constructors of a comonadic modal type theory (S4 neccesity modal type theory) such as that studied in Bierman and de Paiva (1999). Indeed, the fruitfulness of this approach discloses that Montague's intensional logic is, in essence, the marriage of typed higher-order logic and simple type theory with a comonadic modality.

An attractive feature of Montague (1973) is that it provides separate logical forms for de re and de dicto sentences. However, Montague (1973)'s use of λ-notation in the representation of de re sentences means that system does not satisfy the standard reduction rule for application of λ-terms. This problem can be rectified by applying the lessons of Bierman and de Paiva (ibid.), or of Pfenning and Davies (2000). However, the syntax for the intension operator suggested by each of these works is unwieldy. The syntax for the intension operator proposed below is simpler, and avoids further complications to the logic as a whole (such as in ibid.).

A deductive system for the logic is also presented below. As an example of the power of this system, we show that substitution of terms deemed 'rigid designators' (after Kripke 1980) commutes with the intension operator. This fact completes the current work's treatment of binding and substitution into ‘oblique’ (or ‘modal’) contexts, providing a simple and intuitive approach to an issue dating to Quine (1943) and before.

The system **MIL** is presented in Sections 1 and 2. Any substantial differences with Montague’s original are noted, and alternative approaches to comonadic modal type theory are discussed. Section 3 introduces the deductive system and certain theorems thereof.

2.1 Types

The types of **MIL** are summarized in Figure 2.1, below. More detailed remarks follow.

$$\frac{}{\mathsf{E}, \mathsf{T}\ \text{Type}}\ (\text{Basic Types}) \qquad \frac{A\ \text{Type} \qquad B\ \text{Type}}{A \to B\ \text{Type}}\ (\to \text{Form.})$$

$$\frac{A\ \text{Type}}{\flat A\ \text{Type}}\ (\flat\ \text{Form.})$$

Figure 2.1: **MIL** Type Formation Rules

The types are defined recursively by the following rules:

- (T1). There are basic types E, T.

  These do duty for Montague’s type e of ‘entities’ and t of ‘truth values’. We use capital letters for types, in line with the practice of modern type theorists.

- (T2). If A and B are types, then so is $A \to B$.

  The constructor $(- \to -)$ for function types does duty for Montague’s $\langle -, - \rangle$, again using a prevalent modern notation.

- (T3). If A is a type, then so is $\flat A$.

  The type operator $\flat$ does duty for Montague’s $\langle s, - \rangle$ (instead of the analog $(\mathsf{S} \to -)$). We abandon the function type-style notation, since, under the semantics for MIL provided below, $\flat$ is not in general replaceable using $(- \to -)$.

In terms of the Kripke semantics of Montague's system, the difference comes from allowing the set of entities to vary between worlds. Writing $E(w)$ for the set of entities at a world $w \in W$, an intension i will be an assignment

$$w \mapsto i(w) \in E(w) \quad .$$

That is, i is a member of the set $\prod_{w \in W} E(w)$ of dependent functions, functions whose 'codomain' varies across their domain, rather than a member of the function set E^W for a constant set of entities E.

Instead of the function type interpretation, $\flat$ will be interpreted as a comonadic type operator or S4 necessity type modality, as appearing in, inter al., Bierman and de Paiva (1999). The notation $\flat$ for an S4 modality appeared in Awodey, Birkedal, and Scott (2002). Work such as Schreiber (2011) and Shulman (2015) built the convention, used here, that $\flat$ be used for S4 necessity type modalities in particular.

The natural treatment of $\flat$ as a type modality with a very general semantics underscores the wisdom of Montague's decision to use the type operator $\langle s, - \rangle$ rather than integrate a basic type s.

2.2 Terms

The term calculus of **MIL** consists of the rules for intuitionistic typed higher order logic (cf. Jacobs 1999, Lambek and Scott 1988), together with two additional rules for the 'intension' and 'extension' operators, $^\frown$ and $^\smile$, respectively. Predicates are taken to be terms of type $A \to \mathsf{T}$, where A is a type. Rather than taking the necessity modality on predicates, $\Box$, as primitive, it is defined using $^\frown$. This speeds a number of proofs below.

The terms of **MIL** are summarized in Figure 2.2, below. Detailed remarks follow.

$$\frac{}{\Gamma, x : A, \Delta \mid x : A} \qquad\qquad \frac{\cdot \mid c : A}{\Gamma \mid c : A}$$

$$\frac{\Gamma, x : A \mid t : B}{\Gamma \mid \lambda x.t : A \to B}\ (\to \text{Intro.}) \qquad\qquad \frac{\Gamma \mid t : A \to B \qquad \Gamma \mid u : A}{\Gamma \mid tu : B}\ (\to \text{Elim.})$$

$$\frac{\Gamma \mid s_1 : \flat A_1 \qquad \text{...} \qquad \Gamma \mid s_n : \flat A_n \qquad x_1 : \flat A_1, ..., x_n : \flat A_n \mid t(x_1, ..., x_n) : B}{\Gamma \mid {}^\frown t([s_1], ..., [s_n]) : \flat B}\ (\flat \text{ Intro., or Intension})$$

$$\frac{\Gamma \mid t : \flat A}{\Gamma \mid {}^\smile t : A}\ (\flat \text{ Elim., or Extension})$$

$$\frac{\Gamma \mid \phi : \mathsf{T} \qquad \Gamma \mid \psi : \mathsf{T}}{\Gamma \mid \phi \wedge \psi : \mathsf{T}}\ (\wedge \text{ Form.}) \text{, and similarly for } \vee \text{ and } \Rightarrow.$$

$$\frac{\Gamma \mid \phi : \mathsf{T}}{\Gamma \mid \neg\phi : \mathsf{T}}\ (\neg \text{ Form.}) \qquad\qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid u : A}{\Gamma \mid t =_A u : \mathsf{T}}\ (=_A \text{ Form.})$$

$$\frac{}{\cdot \mid \top : \mathsf{T}}\ (\top \text{ Form.}) \text{, and similarly for } \bot.$$

$$\frac{\Gamma, x : A \mid \phi : \mathsf{T}}{\Gamma \mid \forall x.\phi : \mathsf{T}}\ (\forall \text{ Form.}) \text{, and similarly for } \exists.$$

Figure 2.2: Term Calculus of **MIL**

2.2.1 Contexts

Each type A is assigned an infinite set of variables of type A. We write $x : A$ when x is a variable of type A and say that $x : A$ is a typed variable. We write Γ, Δ for a finite or empty lists of typed variables. **MIL** considers terms t together with type B and a list of typed variables Γ containing at least all the free variables in t. Γ is called the context of t. Such terms-in-context are formally written $\Gamma \mid t : B$.[1] For readability, however, the $\Gamma \mid$ may be omitted when inferrable. Schematic lists such as $x_1, ..., x_n$ (where $x_1, ..., x_n$ denote metavariables) may possibly be empty and may be abbreviated as $\mathbf{x}$.

The terms-in-context (henceforth simply terms) are defined recursively the labeled rules:

2.2.2 Typed Higher-Order Logic

Lambda Calculus

- (LC1). There is a term $\Gamma, x : A, \Delta \mid x : A$ for any context $\Gamma, x : A, \Delta$.
- (LC2). If $\cdot \mid c : A$ is a term, then $\Gamma \mid c : A$ is a term.
- (LC3). If $\Gamma, x : A \mid t : B$ is a term, then $\Gamma \mid \lambda x.t : A \to B$ is a term.
- (LC4). If $t : A \to B$ and $u : A$ are terms, then $tu : B$ is a term.

Logic

- (L1). If $\phi : \mathsf{T}$ and $\psi : \mathsf{T}$ are terms, then $\phi \wedge \psi : \mathsf{T}$, $\phi \vee \psi : \mathsf{T}$, and $\phi \Rightarrow \psi : \mathsf{T}$ are terms.
- (L2). If $\phi : \mathsf{T}$ is a term, then $\neg\phi : \mathsf{T}$ is a term.
- (L3). There are terms $\cdot \mid \top : \mathsf{T}$ and $\cdot \mid \bot : \mathsf{T}$.
- (L4). If $\Gamma, x : A \mid \phi : \mathsf{T}$ is a term, then $\Gamma \mid \forall x.\phi : \mathsf{T}$ and $\Gamma \mid \exists x.\phi : \mathsf{T}$ are terms.
- (L5). If $t : A$ and $u : A$ are terms, then $t =_A u : \mathsf{T}$ is a term.

2.2.3 Modal Type Theory

Intuitively, the next rule, *Intension*, takes a term $x_1 : \flat A_1, ..., x_n : \flat A_n \mid t$, prepends a circumflex ($\hat{}$) to it, and marks each free variable x of t by enclosing it in square brackets. For example, a

[1] The symbol $\vdash$ is reserved for the entailment relation on terms of type T, defined below.

term

$$x : \flat A, y : \flat B \mid f(x, g(y))$$

yields via *Intension* the term

$$x : \flat A, y : \flat B \mid {}^\frown f([x], g([y])) \quad .$$

The notation [...] is chosen to suggest substitution, and it indeed indicates a variant of 'explicit substitution' (c.f. Abadi et al. 1991); the appearance of $[s]$ as a substring of a well-formed t indicates that s was substituted for a(n unknown) free variable at the step where a(n inferrable) ${}^\frown$ was adduced.

The precise statement of *Intension* requires an extended discussion of notation, below. The rule is first stated here to complete the statement of the term calculus.

- (Intension). If $x_1 : \flat A_1, ..., x_n : \flat A_n \mid t(x_1, ..., x_n) : B$ and $\Gamma \mid s_1 : \flat A_1, ..., s_n : \flat A_n$ are terms, then $\Gamma \mid {}^\frown t([s_1], ..., [s_n]) : \flat B$ is a term.
- (Extension). If $t : \flat A$ is a term, then ${}^\smile t : A$ is a term.

The rules *Intension* and *Extension* are to be regarded as introduction and elimination rules, respectively, for the constructor $\flat$.

Generalized Substitution

As noted, applying *Intension* to a term t involves marking each free variable x of t with square brackets and prepending a ${}^\frown$. The statement of *Intension* requires a meta-syntactic notation conveying this.

To this end, we generalize the usual notation for substitution by allowing substitutions like $({}^\frown s)[[t]/x]$. The strings ${}^\frown s$ and $[t]$ are not terms of $\mathbf{MIL}$, though they may be introduced by *Intension* as substrings of terms.

Definition 1 (Generalized Substitution)**.** *Assume terms s and $t_0, ..., t_n$, variables $x_0, ..., x_n$, and strings $h \in \{s, {}^\frown s\}$ and $b_i \in \{t_i, [t_i]\}$ $(0 \leqslant i \leqslant n)$. Then $h[\mathbf{b}/\mathbf{x}]$ is defined by:*

- $y[\mathbf{b}/\mathbf{x}] :\equiv \begin{cases} y & y \notin \{x_0, ...x_n\} \\ b_i & y \equiv x_i \end{cases}$
- $c[\mathbf{b}/\mathbf{x}] :\equiv c$
- $(\lambda y.r)[\mathbf{b}/\mathbf{x}] :\equiv \begin{cases} \lambda y.(r[\mathbf{b}/\mathbf{x}]) & y \notin \{x_0, ...x_n\} \\ \lambda y.(r[b_0, ...b_{i-1}, b_{i+1}, ...b_n/x_0, ...x_{i-1}, x_{i+1}, ...x_n]) & y \equiv x_i \end{cases}$
- $qr[\mathbf{b}/\mathbf{x}] :\equiv q[\mathbf{b}/\mathbf{x}]r[\mathbf{b}/\mathbf{x}]$
- $(\phi \wedge \psi)[\mathbf{b}/\mathbf{x}] :\equiv \phi[\mathbf{b}/\mathbf{x}] \wedge \psi[\mathbf{b}/\mathbf{x}]$*, and similarly for* $\vee, \Rightarrow$.
- $(\neg\phi)[\mathbf{b}/\mathbf{x}] :\equiv \neg(\phi[\mathbf{b}/\mathbf{x}])$
- $\top[\mathbf{b}/\mathbf{x}] :\equiv \top$*, and similarly for* $\bot$.
- $(\forall y.\phi)[\mathbf{b}/\mathbf{x}] :\equiv \begin{cases} \forall y.(\phi[\mathbf{b}/\mathbf{x}]) & y \notin \{x_0, ...x_n\} \\ \forall y.(\phi[b_0, ...b_{i-1}, b_{i+1}, ...b_n/x_0, ...x_{i-1}, x_{i+1}, ...x_n]) & y \equiv x_i \end{cases}$*, and similarly for* $\exists$.
- $(q =_A r)[\mathbf{b}/\mathbf{x}] :\equiv q[\mathbf{b}/\mathbf{x}] =_A r[\mathbf{b}/\mathbf{x}]$
- $[r][\mathbf{b}/\mathbf{x}] :\equiv [r[\mathbf{b}/\mathbf{x}]]$
- $(\hat{}\,r)[\mathbf{b}/\mathbf{x}] :\equiv \hat{}\,(r[\mathbf{b}/\mathbf{x}])$
- $(\check{}\,r)[\mathbf{b}/\mathbf{x}] :\equiv \check{}\,(r[\mathbf{b}/\mathbf{x}])$

Substitution in $\mathbf{MIL}$ is simply generalized substitution in which $h, \mathbf{b}$ are required to be terms.

The reader may choose to view generalized substitution as ordinary substitution within an auxiliary language $\mathbf{MIL}' \supset \mathbf{MIL}$.

The language $\mathbf{MIL}'$ is exactly the same as $\mathbf{MIL}$, except the clause *Intension* is replaced by the following:

- $(\mathbf{MIL}'1)$. If t is a term, then $[t]$ is a term.
- $(\mathbf{MIL}'2)$. If $t : B$ is a term, then $\hat{}\,t : \flat B$ is a term.

With the notation now established, we review the rule *Intension*.

- (Intension). If $x_1 : \flat A_1, ..., x_n : \flat A_n \mid t(x_1, ..., x_n) : B$ and $\Gamma \mid s_1 : \flat A_1, ..., s_n : \flat A_n$ are terms, then $\Gamma \mid {}^\frown t([s_1], ..., [s_n]) : \flat B$ is a term.

Convention 2. *We may abbreviate $t([s_1], ..., [s_n])$ by $t([\mathbf{s}])$, or even $t[\mathbf{s}]$.*

Convention 3. *We may write $\flat\Gamma$ for a finite or empty list of typed variables with types all of the form $\flat A$ (such as $x_1 : \flat A_1, ..., x_n : \flat A_n$).*

2.2.4 Defined Notations

- (Box). If $\mathbf{x} : \flat\mathbf{A} \mid \phi(\mathbf{x}) : \mathsf{T}$ is a term, then $\mathbf{x} : \flat\mathbf{A} \mid \Box\phi([x_1], ..., [x_n]) : \mathsf{T}$ is defined by

$$\Box\phi([x_1], ..., [x_n]) :\equiv ({}^\frown\top =_{\flat\mathsf{T}} {}^\frown\phi([x_1], ..., [x_n])) \quad .$$

  Remark 4. *Intuitively, $\Box\phi([\mathbf{x}])$ 'checks whether ϕ has the same intension as $\top$'. This reduction of $\Box$ using ${}^\frown$ will simplify several proofs about* **MIL** *below. A similar reduction appears as early as Montague's* Universal Grammar *(1970).*

- (Comultiplication, or Modal Principle 4). If $\flat\Gamma \mid x : \flat A$, then $\flat\Gamma \mid \delta([x]) : \flat\flat A$ is defined by

$$\delta([x]) :\equiv {}^\frown[x] \quad .$$

- (Functorial Action of $\flat$). If $\mathbf{x} : \mathbf{A} \mid t(\mathbf{x}) : B$, then $\mathbf{x} : \flat\mathbf{A} \mid \flat t([x_1], ..., [x_n]) : \flat B$ is defined by

$$\flat t([x_1], ..., [x_n]) :\equiv {}^\frown t({}^\smile[x_1], ..., {}^\smile[x_n]) \quad .$$ [2]

  Remark 5. *In addition to maintaining the balance between 'intensional operators' ${}^\frown, \Box, \flat, \delta$ and their corresponding square brackets, explicitly tracking square brackets with $\flat$ will also serve to enforce a distinction between the terms $\flat(t(\mathbf{s}[\mathbf{y}]))$ (where $\mathbf{s}(\mathbf{y}) : \mathbf{A}$) and $\flat t[s(\mathbf{y})]$ (where $\mathbf{s}(\mathbf{y}) : \flat\mathbf{A}$), which would otherwise both be written $\flat(t(\mathbf{s}(\mathbf{y})))$.*

[2] We include the $n = 0$ case: given a constant $\cdot \mid c : A$, then $\cdot \mid \flat c \equiv {}^\frown c[\cdot/\cdot] \equiv {}^\frown c : \flat A$.

2.2.5 De Re and De Dicto

The merit of the current approach, with its particular formulation of *Intension* and *Box*, can be seen by considering the intended translations into MIL of sentences such as

$$\text{The president is necessarily the commander-in-chief.} \tag{2.1}$$

Such a sentence is generally held to have two readings: a de re and a de dicto. Under the de re construal, 2.1 asserts that Donald Trump (the current president) must be the commander in chief. Put another way, the property "is necessarily commander-in-chief" is predicated *of the thing* (Latin *de re*) referenced by "the president". Under the de dicto reading, 2.1 asserts that the positions of president and commander-in-chief must be occupied by the same person. There is an intuition that such a relation holds due to a relation in the definitions *of the words* "president" and "commander-in-chief" (cf. Latin *de dicto*, meaning "of the utterance"). These readings are truly different, for the former is false (as Trump was not always the commander-in-chief) while it is quite plausible that the latter is true (command of the armed forces arguably being an integral part of being president).

A key achievement of Montague, and a desideratum for subsequent work, is to supply logical forms for both de re and de dicto readings. A number of approaches to this are now available to us, and are surveyed below. Montague's groundbreaking approach is not entirely satisfactory, as he abandons the usual rules for λ-calculus. The current work resolves this issue. The type theories of Bierman and de Paiva (1999) and Pfenning and Davies (2000) are also observed to supply a solution.

Montague's Approach

In Montague (1973) (modified to the current notation), if "The President" translates to a constant $\cdot \mid p : \flat E$ and "is the commander-in-chief" to a predicate $x : \flat E \mid c(x) : \mathsf{T}$, then the sentence is translated under the de dicto construal to

$$\cdot \mid \Box c(p) : \mathsf{T} \quad ,$$

and under the de re one to

$$\cdot \mid (\lambda x.\Box c(x))p : \mathsf{T} \quad .$$

Since these two terms are intended to be distinct, the usual reduction

$$(\lambda x.\Box s(x))u \rightsquigarrow_\beta \Box s(u)$$

must fail.

The Current Approach

By contrast, the de dicto reading is represented in **MIL** by

$$\cdot \mid \Box c(p) : \mathsf{T}$$

and the de re by

$$\cdot \mid \Box c([p]) : \mathsf{T} \quad .$$

These two terms will not be equal in general: substitution of a term p commutes with *Box* (or *Intension*) only when p is a *rigid designator* (in the sense of Theorem 15). There is consequently no need to deviate from the usual rules for function types.

Other Approaches

Works such as Bierman and de Paiva (1999) and Pfenning and Davies (2000) contain operations analogous to *Intension*.

Figure 22 summarizes the available logical forms for de re and de dicto. The columns labeled "As Proposed" give de re and de dicto forms as originally proposed by the works cited. The columns labeled "Adapted" adopt some of the syntactic conventions of the current work, facilitating a more direct comparison. In particular the box of Bierman and de Paiva (1999) and Pfenning and Davies (2000) is replaced by $\hat{}$, harmonizing the syntax for the intension operator. Since there is little difference in the treatment of de dicto, the key comparison is provided by the "Adapted: De Re" column (starred).

	Syntax for Intension Operator			
	As Proposed		**Adapted**	
Reference	**De Dicto**	**De Re**	**De Dicto**	**⋆De Re⋆**
Montague (1973)	$\hat{}t(u)$	$\lambda x.\hat{}t(x)(u)$	$\hat{}t(u)$	$(\lambda x.\hat{}t(x))(u)$
Bierman and de Paiva (1999)	$\mathsf{box}\ t(u)\ \mathsf{with}\ \cdot\ \mathsf{for}\ \cdot$	$\mathsf{box}\ t(x)\ \mathsf{with}\ u\ \mathsf{for}\ x$	$\hat{}t(u)\ \mathsf{with}\ \cdot\ \mathsf{for}\ \cdot$	$\hat{}t(x)\ \mathsf{with}\ u\ \mathsf{for}\ x$
Pfenning and Davies (2000)	$\mathsf{box}\ t(u)$	$\mathsf{let\ box}\ x = u\ \mathsf{in\ box}\ t(x)$	$\hat{}t(u)$	$\mathsf{let}\ \hat{}x = u\ \mathsf{in}\ \hat{}t(x)$
Current Work	$\hat{}t(u)$	$\hat{}t([u])$	$\hat{}t(u)$	$\hat{}t([u])$

Figure 2.3: Comparison of Comonadic Modal Type Theories

All three of Bierman and de Paiva (1999), Pfenning and Davies (2000), and the present work have β-reduction, thus resolving the issue from Montague (1973). The syntax of the present work has the advantage of being simpler than the alternatives, however. This is particularly evident under nested intension operators. For instance, the term rendered

$$\hat{}t(x)\ \mathsf{with}\ (\hat{}s(y)\ \mathsf{with}\ r\ \mathsf{for}\ y)\ \mathsf{for}\ x : \flat A$$

in Bierman and de Paiva (adapted syntax) is rendered in **MIL** as the more manageable

$$\hat{}t[\hat{}s[r]] : \flat A \quad .$$

This has advantages in calculations such as those in Section 2.5.

On the other hand, **MIL** does not obey the subformula property ($^\frown t(x)$ may appear in the derivation of $^\frown t[^\frown s[r]]$, but does not appear as a subformula).

2.2.6 The Restriction on Contexts

The restriction to contexts of the form $x_1 : \flat A_1, ..., x_2 : \flat A_2$ in the rules *Intension* and *Box* will permit (in the next chapter) a very general semantics for **MIL** which includes using the '(co)Kleisli lift' operation of a comonad to interpret *Intension*.

Similar regimentations on the context are employed in the comonadic type theories of Bierman and de Paiva (1999) and Pfenning and Davies (2000). Strikingly, while Montague (1973) does not restrict contexts for $^\frown$ and $\Box$, its semantic analysis of English is regimented so that more general contexts for $^\frown$ and $\Box$ are not employed.

It is possible to do away with any syntactic restriction on contexts if we restrict how contexts are interpreted (see Chapter 3). Montague (1973) in effect takes this tack by restricting to a constant domain of entities (see Chapter 4).

2.2.7 Tense Modalities

Montague's non-S4 tense modalities are not treated here. A semantics is nevertheless indicated in Chapter 4.

2.3 Structural Rules

The structural rules (weakening, contraction, permutation, and cut) are not directly included in the deductive system, as they can be derived by routine induction on the structure of t:

Lemma 6 (Weakening). $\dfrac{\Gamma, \Delta \mid t : B}{\Gamma, x : A, \Delta \mid t : B}$

Lemma 7 (Contraction). $\dfrac{\mathbf{x} : \mathbf{A} \mid t : B \qquad A_i = A_j}{x_1 : A_1, ..., x_{j-1} : A_{j-1}, x_i : A_i, x_{j+1} : A_{j+1}, ..., x_n : A_n \mid t : B}$

Lemma 8 (Permutation)**.**

$$\frac{\mathbf{x} : \mathbf{A} \mid t : B}{x_1 : A_1, ..., x_{i-1} : A_{i-1}, x_j : A_j, x_{i+1} : A_{i+1}, ..., x_{j-1} : A_{j-1}, x_i : A_i, x_{j+1} : A_{j+1}, ...x_n : A_n \mid t : B}$$

Lemma 9 (Substitution)**.** $\dfrac{\mathbf{x} : \mathbf{A} \mid s : B_i \qquad \mathbf{y} : \mathbf{B} \mid t : C}{y_1 : B_1, ..., y_{i-1} : B_{i-1}, \mathbf{x} : \mathbf{A}, y_{i+1} : B_{i+1}, ..., y_n : B_n \mid t[s/y_i] : C}$

Despite Lemma 9, MIL does not have the subformula property, as mentioned in Section 2.2.5.

2.4 Entailment

We complete the deductive system by axiomatizing an entailment relation, $\vdash$, on the terms of type T. For comparison, a deductive calculus for Montague (1973) is presented in Gallin (1975).

2.4.1 Higher-Order Logic

The following axioms are simply those of intuitionistic typed higher-order logic (cf. Jacobs 1999, Lambek and Scott 1988).

1. $\phi \vdash \phi$
2. $\dfrac{\phi \vdash \psi \qquad t : A}{\phi[t/x] \vdash \psi[t/x]}$, for $x : A$ (and similarly for simultaneous substitution, if desired).
3. $\dfrac{\phi \vdash \psi \qquad \psi \vdash \theta}{\phi \vdash \theta}$
4. $\top \vdash x =_A x$, for any $x : A$.
5. $\phi \wedge x =_A x' \vdash \phi[x'/x]$, for any $x, x' : A$.
6. $\forall x.f(x) =_B g(x) \dashv\vdash f =_{(A \to B)} g$, for any $x : A$ and $f, g : A \to B$.
7. $\phi \Leftrightarrow \psi \dashv\vdash \phi =_{\mathsf{T}} \psi$, for any $\phi, \psi : \mathsf{T}$.
8. $\top \vdash (\lambda x.t)x' =_B t[x'/x]$, for any $t : B, x' : A$.
9. $\top \vdash (\lambda x.f(x)) =_{(A \to B)} f$, for any $f : A \to B, x : A$.
10. $\phi \vdash \top$

11. $\bot \vdash \phi$

12. $\phi \vdash \psi \wedge \theta$ iff $\phi \vdash \psi$ and $\phi \vdash \theta$.

13. $\phi \vee \psi \vdash \theta$ iff $\phi \vdash \theta$ and $\psi \vdash \theta$.

14. $\phi \vdash \psi \Rightarrow \theta$ iff $\phi \wedge \psi \vdash \theta$.

15. $\Gamma \mid \exists x.\phi \vdash \psi$ iff $\Gamma, x : A \mid \phi \vdash \psi$.

16. $\Gamma \mid \phi \vdash \forall x.\psi$ iff $\Gamma, x : A \mid \phi \vdash \psi$.

2.4.2 S4 Modal Logic

The following are the standard S4 axioms, adapted to the present approach. The author consulted Awodey et al. (2014).

(ML1). $\dfrac{\mathbf{x} : \flat\mathbf{A} \mid \phi(\mathbf{x}) \vdash \psi(\mathbf{x})}{\mathbf{x} : \flat\mathbf{A} \mid \Box\phi([\mathbf{x}]) \vdash \Box\psi([\mathbf{x}])}$

(ML2). $\mathbf{x} : \flat\mathbf{A} \mid \Box\phi([\mathbf{x}]) \vdash \phi(\mathbf{x})$

(ML3). $\mathbf{x} : \flat\mathbf{A} \mid \Box\phi([\mathbf{x}]) \vdash \Box\Box\phi([[\mathbf{x}]])$

(ML4). $\mathbf{x} : \flat\mathbf{A} \mid \Box\phi([\mathbf{x}]) \wedge \Box\psi([\mathbf{x}]) \vdash \Box(\phi \wedge \psi)([\mathbf{x}])$

(ML5). $\mathbf{x} : \flat\mathbf{A} \mid \top \vdash \Box\top$

2.4.3 S4 Modal Type Theory

The following axioms are suggested by the intended semantics (cf. Manes 1976, axioms for a Kleisli triple). The principle *MTT2* appears in Gallin (1975).

(MTT1). $\top \vdash \hat{}\check{}[x] =_{\flat A} x$, for all variables $x : \flat A$.

(MTT2). $\top \vdash \check{}\hat{}t([\mathbf{x}]) =_{A} t(\mathbf{x})$, for all $\mathbf{x} : \flat\mathbf{A} \mid t : A$.

(MTT3). $\top \vdash \hat{}t([\hat{}\mathbf{s}([\mathbf{x}])]) =_{\flat C} \hat{}t(\hat{}s_1([[\mathbf{x}]], ..., \hat{}s_n([[\mathbf{x}]])$, where $\mathbf{x} : \flat\mathbf{A} \mid s_i(\mathbf{x}) : B_i$, for each i $(0 \leqslant i \leqslant n)$, and $\mathbf{y} : \flat\mathbf{B} \mid t(\mathbf{y}) : C$.

Convention 10. *We may write '$\Gamma \mid \phi$ holds' for an assertion of* $\Gamma \mid \top \vdash \phi$.

Definition 11. *A **theory** of* MIL *consists of a set* $\mathbf{T} = \mathbf{T_c} \bigcup \mathbf{T_e}$*, where* $\mathbf{T_c}$ *is a set of constants* $\{c_1 : A_1, c_2 : A_n, ...\}$ *and* $\mathbf{T_e}$ *is a set of entailments* $\{\Gamma_1 \mid \phi_1 \vdash \psi_1, \Gamma_2 \mid \phi_2 \vdash \psi_2, ...\}$.

2.5 Derivable Principles

Proposition 12. *The following principles are derivable:*

- (Necessitation).
$$\frac{\mathbf{x} : \flat\mathbf{A} \mid \top \vdash \phi(\mathbf{x})}{\mathbf{x} : \flat\mathbf{A} \mid \top \vdash \Box\phi([\mathbf{x}])}$$
- (K).
$$\mathbf{x} : \flat\mathbf{A} \mid \Box(\phi \Rightarrow \psi)([\mathbf{x}]) \vdash \Box\phi([\mathbf{x}]) \Rightarrow \Box\psi([\mathbf{x}])$$
- (Converse Barcan Principle).
$$\frac{\mathbf{x} : \flat\mathbf{A}, y : \flat B \mid \phi(\mathbf{x}, y) : \mathsf{T}}{\mathbf{x} : \flat\mathbf{A} \mid \Box(\forall y.\phi)([\mathbf{x}]) \vdash \forall y.\Box\phi([\mathbf{x}, y])}$$
- (Persistence of Identity).
$$x : \flat A, x' : \flat A \mid x =_{\flat A} x' \vdash \Box([x] =_{\flat A} [x'])$$

Proof.

- (Necessitation). Follows using *ML5* and *ML1*.
- (K). Follows using *ML4* and *ML1*.
- (Converse Barcan Principle). Follows using *ML1*.
- (Persistence of Identity). Follows using Necessitation.

□

Lemma 13 ($\flat$ Preserves Composition)**.** *Given* $y_1 : B_1, ..., y_n : B_n \mid t(\mathbf{y}) : C$ *and* $\mathbf{x} : \mathbf{A} \mid s_1(\mathbf{x}) : B_1, ..., \mathbf{x} : \mathbf{A} \mid s_n(\mathbf{x}) : B_n$*, the equality*

$$\mathbf{x} : \flat\mathbf{A} \mid \flat(t(\mathbf{s}[\mathbf{x}])) =_{\flat C} \flat(t[\flat(\mathbf{s}[\mathbf{x}])])$$

holds.

Proof.

$$\begin{aligned}
\flat(t(\mathbf{s}[\mathbf{x}])) &\equiv \hat{}(t(\mathbf{s}(\check{}[\mathbf{x}])) \\
&=_{\flat C} \hat{}t(\check{}\hat{}\mathbf{s}(\check{}[[\mathbf{x}]])) && (MTT2) \\
&=_{\flat C} \hat{}t(\check{}[\hat{}\mathbf{s}(\check{}[\mathbf{x}])]) && (MTT3) \\
&\equiv \flat(t[\flat(\mathbf{s}[\mathbf{x}])])
\end{aligned}$$

□

Lemma 14. *Given* $\mathbf{x} : \flat\mathbf{A} \mid t(\mathbf{x}) : B$*, the equality*

$$\mathbf{x} : \flat\mathbf{A} \mid \flat t[\delta[x_1], ..., \delta[x_n]] =_{\flat B} \hat{}t[\mathbf{x}]$$

holds.

Proof.

$$\begin{aligned}
\flat t[\delta[x_1], ..., \delta[x_n]] &\equiv \hat{}t(\check{}[\delta[x_1]], ..., \check{}[\delta[x_n]]) \\
&\equiv \hat{}t(\check{}[\hat{}[x_1]], ..., \check{}[\hat{}[x_n]]) \\
&=_{\flat C} \hat{}t(\check{}\hat{}[[x_1]], ..., \check{}\hat{}[[x_n]]) && (MTT3) \\
&=_{\flat C} \hat{}t([x_1], ..., [x_n]) && (MTT2)
\end{aligned}$$

□

Theorem 15 (Substitution of Rigid Designators in Modal Contexts)**.** *Let*

$$x_1 : \flat A_1, ..., x_m : \flat A_m \mid s_i(\mathbf{x}) : \flat B_i$$

for each i $(1 \leqslant i \leqslant n)$*. Then '*$\hat{}$*commutes with substitution of* $\mathbf{s}$*' iff '*$\mathbf{s}$ *respects* δ*'. That is,*

$$(\flat s_i[\delta_{A_1}[x_1], ..., \delta_{A_m}[x_m]]) =_{\flat\flat B} \delta_{B_i}[s_i] \textit{ for all } i$$

iff

$$(\hat{}\,t[\mathbf{s}(\mathbf{x})]) =_{\flat C} \hat{}\,t(\mathbf{s}[\mathbf{x}]),\ \textit{for each } y_1 : \flat B_1, ..., y : n : \flat B_n \mid t : C \quad .$$

Proof. ($\Rightarrow$). Assume $(\flat s_i[\delta_{A_1}[x_1], ..., \delta_{A_m}[x_m]]) =_{\flat\flat B} \delta_{B_i}[s_i]$ for each i. Then

$$\begin{aligned}
(\hat{}\,t[\mathbf{s}]) &=_{\flat C} \flat t[\delta[s_1], ..., \delta[s_n]] && \text{(Lemma 14)}\\
&=_{\flat C} \flat t[\flat s_1[\delta[x_1], ..., \delta[x_m]], ..., \flat s_n[\delta[x_1], ..., \delta[x_m]]] && \text{(Assumption)}\\
&\equiv \flat t[\flat \mathbf{s}[\delta[x_1], ..., \delta[x_m]]] && \text{(Associativity of substitution)}\\
&=_{\flat C} \flat(t(\mathbf{s}[\delta[x_1], ..., \delta[x_m]])) && \text{(Lemma 13)}\\
&=_{\flat C} \hat{}\,t(\mathbf{s}[\mathbf{x}]) && \text{(Lemma 14)}
\end{aligned}$$

($\Leftarrow$). Assume $(\hat{}\,t[\mathbf{s}(\mathbf{x})]) =_{\flat C} \hat{}\,t(\mathbf{s}[\mathbf{x}])$. Then

$$\begin{aligned}
\flat s_i[\delta_{A_1}[x_1], ..., \delta_{A_m}[x_m]] &=_{\flat\flat B} \hat{}\,s_i[\mathbf{x}] && \text{(Lemma 14)}\\
&\equiv \hat{}\,(y_i(\mathbf{s}[\mathbf{x}]))\\
&=_{\flat\flat B} (\hat{}\,[y_i])(\mathbf{s}(\mathbf{x})) && \text{(Assumption)}\\
&\equiv \delta[s_i]
\end{aligned}$$

□

Corollary 16. *Let* $x_1 : \flat A_1, ..., x_m : \flat A_m \mid s_i : \flat B_i$ *for each* i $(1 \leqslant i \leqslant n)$*. Assume '*s *respects* δ*', i.e.*

$$(\flat s_i[\delta_{A_1}[x_1], ..., \delta_{A_m}[x_m]]) =_{\flat\flat B} \delta_{B_i}[s_i]\ \textit{for all } i \quad .$$

Then, for each $y_1 : \flat B_1, ..., y_n : \flat B_n \mid \phi : \mathsf{T}$*,*

$$\Box\phi[\mathbf{s}(\mathbf{x})] =_{\mathsf{T}} \Box\phi(\mathbf{s}[\mathbf{x}]) \quad .$$

Proof. Assume $(\flat s_i[\delta_{A_1}[x_1], ..., \delta_{A_m}[x_m]]) =_{\flat\flat B} \delta_{B_i}[s_i]$. Then, by the preceding Theorem 15, $(\hat{}t[\mathbf{s}(\mathbf{x})]) =_{\flat C} \hat{}t(\mathbf{s}[\mathbf{x}])$. Resultingly,

$$\begin{aligned}
\Box\phi[\mathbf{s}(\mathbf{x})] &\equiv \hat{}\top =_{\flat\mathsf{T}} \hat{}\phi[\mathbf{s}(\mathbf{x})] \\
&=_{\mathsf{T}} \hat{}\top =_{\flat\mathsf{T}} \hat{}\phi(\mathbf{s}[\mathbf{x}]) && \text{(Assumption)} \\
&\equiv \Box\phi(\mathbf{s}[\mathbf{x}])
\end{aligned}$$

$\square$

Chapter 3

Categorical Semantics of MIL

Below, we develop a categorical semantics for the system MIL of the previous chapter. This combines the categorical semantics for predicate S4 modal logic (first formulated by Reyes and Zolfaghari 1991) and the semantics of simple type theory with a comonadic modality (Bierman and de Paiva 1999). The result is an interpretation of MIL using a comonad on a topos. This interpretation represents a slightly simplified version of the semantics of Awodey, Buchholtz, and Zwanziger (2016).

This framework abstracts away from contingent features of the traditional Kripke and Boolean-valued semantics for Montague's logic, such as the use of possible worlds. The presence of a comonad turns out to be the essential, common structure in these examples. The concept of a monad has recently been employed in natural language semantics in order to modularly integrate more complex semantic representations while maintaining the principle of compositionality (c.f. Shan 2002, Barker 2002). Comonads can serve a similar function, though with a different flavor. The implicit presence of a comonad in MIL suggests that they are as fundamental to natural language semantics as are monads.

Section 1 recalls key facts concerning comonads. Section 2 gives the categorical semantics of MIL, and Section 3 proves the soundness of the deductive calculus of Chapter 2 with respect to that semantics.

3.1 Categorical Preliminaries

Before proceeding to interpret of MIL, we review the required categorical notions.

In what follows, we will freely reference the structures existing in any topos that are used in the standard semantics of higher order logic. Definitions for such structures are provided only in the context of the examples. The general definitions are available in, inter al., Lambek and Scott (1986), Mac Lane and Moerdijk (1992), and Johnstone (2002, Part D).

3.1.1 Comonads

In the rest of the current work, we will make extensive use of the comonad concept.

Comonadic Modeling

Comonads (and their dual, monads) arose as fundamental structures in category theory. Since Moggi (1991), however, monads have also played a significant role in functional programming, and now natural language semantics (Shan 2002, Barker 2002). In natural language semantics (as in functional programming), monads offer a way of enriching semantic representations without losing compositionality. For instance, one may wish to take into account not just the truth-conditional content of expressions, but also the effect of their use on a given listener's knowledge state. By contrast (at least in computer science) comonads have potential to regiment a semantics involving "context-dependence" (Uustalu and Vene 2008), that is, extraneous inputs rather than extraneous outputs. This is exactly what we see in MIL, where terms depend on 'intensional' types (of type $\flat A$, say), not just 'extensions' (of type A). It was one of Montague's key insights that the meaning of phrases such as "former senator" is calculated not based on the extension of "senator"— the set of those currently serving as senator—but rather on the intension of "senator"—the information about who served as senator at any given time. In MIL, "former" is thus interpreted as a term of type $\flat(\mathsf{E} \to \mathsf{T}) \to \mathsf{E} \to \mathsf{T}$, rather than $(\mathsf{E} \to \mathsf{T}) \to \mathsf{E} \to \mathsf{T}$.

From Comonads to Kleisli Cotriples

We first recall the categorical definition of a comonad, then delineate its relation to the "Kleisli Cotriple" definition more conventional in computer science.

Definition 17. *A **comonad** on a category* $\mathbf{C}$ *consists of a functor* $\flat : \mathbf{C} \to \mathbf{C}$ *and natural transformations* $\varepsilon : \flat \to \mathrm{id}_{\mathbf{C}}$ *and* $\delta : \flat \to \flat\flat$ *such that, for each* $A \in \mathbf{C}$*, the diagrams of Figures 3.1 and 3.2 commute.*

$$\begin{tikzcd} & \flat A \arrow[dl, "\mathrm{id}_{\flat A}"'] \arrow[d, "\delta_A"] \arrow[dr, "\mathrm{id}_{\flat A}"] & \\ \flat A & \flat\flat A \arrow[l, "\varepsilon_{\flat A}"] \arrow[r, "\flat\varepsilon_A"'] & \flat A \end{tikzcd}$$

Figure 3.1: The Left Counit Law (Lefthand Triangle) and the Right Counit Law (Righthand Triangle)

$$\begin{tikzcd} \flat A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & \flat\flat A \arrow[d, "\flat\delta_A"] \\ \flat\flat A \arrow[r, "\delta_{\flat A}"'] & \flat\flat\flat A \end{tikzcd}$$

Figure 3.2: Coassociativity Law

In what follows, we will only be concerned with comonads whose underlying functors preserve finite limits. Such functors are called **left exact** or **finite limit-preserving**. Henceforth, $\flat$ is assumed to be such a finite limit-preserving comonad. The isomorphism $\langle \flat\pi_1, ..., \flat\pi_n \rangle^{-1}$: $\flat A_1 \times ... \times \flat A_n \cong \flat(A_1 \times ... \times A_n)$ may be denoted by $\phi_{\mathbf{A}}$.[1] With context, we may write simply ϕ.

[1]We include the '$n = 0$' case: the isomorphism $!^{-1} : 1 \to \flat 1$ may be denoted by $\phi.$ (where $\cdot$ denotes the empty list, as always).

In order to interpret the MIL rule *Intension*, we introduce a defined operation, the 'Kleisli lift', on morphisms with domain of the form $\flat A_1 \times ... \times \flat A_n$. In computer science, it is more common to *define* a comonad in terms of the counit and Kleisli lift and axioms. Here, we instead check that these axioms follow from our definition.

Definition 18 (Kleisli Lift, c.f. Manes 1976)**.** *Let* $\flat A_1 \times ... \times \flat A_n \xrightarrow{f} B$ $(n \geqslant 0)$. *Then*

$$\flat A_1 \times ... \times \flat A_n \xrightarrow{f^*} \flat B \quad ,$$

the **Kleisli lift** *of f, is given by the composite*

$$\flat A_1 \times ... \times \flat A_n \xrightarrow{\delta_{A_1} \times ... \times \delta_{A_n}} \flat\flat A_1 \times ... \times \flat\flat A_n \overset{\phi}{\cong} \flat(\flat A_1 \times ... \times \flat A_n) \xrightarrow{\flat f} \flat B \quad .$$

The following lemma will be called upon to prove soundness. Its proof is standard, but most details are nonetheless indicated.

Lemma 19 (Kleisli Cotriple 'Axioms', c.f. Manes 1976)**.**

(KC1). $\varepsilon_A^* = \mathrm{id}_{\flat A}$ *for each* $A \in \mathcal{E}$.

(KC2). $\varepsilon_B f^* = f$ *for each* $f : \flat A_1 \times ... \times \flat A_n \to B$.

$$\begin{tikzcd} \flat A_1 \times ... \times \flat A_n \arrow[rr, "f^*"] \arrow[drr, "f"'] & & \flat B \arrow[d, "\varepsilon_B"] \\ & & B \end{tikzcd}$$

Figure 3.3: Cotriple Axiom 2

(KC3). $g^*\langle f_1^*, ..., f_n^*\rangle = (g\langle f_1^*, ..., f_n^*\rangle)^*$ *for each* $f_i : \flat A_1 \times ... \times \flat A_m \to B_i$ $(1 \leqslant i \leqslant n)$ *and* $g : \flat B_1 \times ... \times \flat B_n \to C$.

Proof.

1. ($\varepsilon_A^* = \mathrm{id}_{\flat A}$). The desired equality follows straightforwardly from the Right Counit Law. It is established as follows:

$$\begin{aligned}
\varepsilon_A^* &= \flat\varepsilon_A \circ \phi \circ \delta_A \\
&= \flat\varepsilon_A \circ \delta_A && (\phi = \mathrm{id}) \\
&= \mathrm{id}_{\flat A} && \text{(Right Counit Law)}
\end{aligned}$$

2. ($\varepsilon_B f^* = f$). The proof makes use of the Left Counit Law. Before proving the desired equality for arbitrary n, the more perspicuous proof for $n = 1$ is presented for reference. Given $f : \flat A \to B$, we have:

$$\begin{aligned}
\varepsilon_B \circ f^* &= \varepsilon_B \circ \flat f \circ \phi \circ \delta_A \\
&= \varepsilon_B \circ \flat f \circ \delta_A && (\phi = \mathrm{id}) \\
&= f \circ \varepsilon_{\flat A} \circ \delta_A && (\text{Naturality of } \varepsilon) \\
&= f && \text{(Left Counit Law)}
\end{aligned}$$

Note that, due to product-preservation, any arrow $\flat A_1 \times ... \times \flat A_n \xrightarrow{f} B$ is 'essentially' an arrow with form $\flat X \to B$, namely

$$\flat(A_1 \times ... \times A_n) \overset{\phi^{-1}}{\cong} \flat A_1 \times ... \times \flat A_n \xrightarrow{f} B \quad .$$

Though daunting, the proof for arbitrary n is, 'essentially', a reduction to the proof for $n = 1$, applied to this $f\phi^{-1}$.

It will make use of the following fact:

Sublemma 20. *The following diagram commutes:*

$$
\begin{tikzcd}
\flat A_1 \times \ldots \times \flat A_n, \arrow[r, "\delta_{A_1} \times \ldots \times \delta_{A_n}"] \arrow[dd, "\langle \flat\pi_{A_1}, \ldots, \flat\pi_{A_n} \rangle^{-1} (= \phi_{\mathbf{A}})"'] & \flat\flat A_1 \times \ldots \times \flat\flat A_n & \\
 & & \flat(\flat A_1 \times \ldots \times \flat A_n) \arrow[ul, bend right, "\langle \flat\pi_{\flat A_1}, \ldots, \flat\pi_{\flat A_n} \rangle (= \phi^{-1}_{\flat A_1, \ldots, \flat A_n})"'] \\
\flat(A_1 \times \ldots \times A_n) \arrow[r, "\delta_{A_1 \times \ldots \times A_n}"] & \flat\flat(A_1 \times \ldots \times A_n) \arrow[uu, "\langle \flat\flat\pi_{A_1}, \ldots, \flat\flat\pi_{A_n} \rangle"] \arrow[ur, bend right, "\flat\langle \flat\pi_{A_1}, \ldots, \flat\pi_{A_n} \rangle (= \flat(\phi^{-1}_{\mathbf{A}}))"'] &
\end{tikzcd}
$$

Proof. Diagram chase using the naturality of δ and the product-preservation of $\flat$. $\square$

We return to the main proof.

$$\varepsilon_B \circ f^* = \varepsilon_B \circ \flat f \circ \phi_{\flat\mathbf{A}} \circ (\delta_{A_1} \times \ldots \times \delta_{A_n})$$

$$= f \circ \varepsilon_{\flat A_1 \times \ldots \times \flat A_n} \circ \phi_{\flat\mathbf{A}} \circ (\delta_{A_1} \times \ldots \times \delta_{A_n}) \quad \text{(Naturality of } \varepsilon\text{)}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \phi_{\flat\mathbf{A}} \circ (\delta_{A_1} \times \ldots \times \delta_{A_n}) \quad \text{(Naturality of } \varepsilon, \phi \text{ iso)}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \phi_{\flat\mathbf{A}} \circ \langle \flat\flat\pi_{A_1}, \ldots, \flat\flat\pi_{A_n} \rangle \circ \delta_{A_1 \times \ldots \times A_n} \circ \langle \flat\pi_{A_1}, \ldots, \flat\pi_{A_n} \rangle^{-1}$$

(Sublemma 20)

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \phi_{\flat\mathbf{A}} \circ \langle \flat\flat\pi_{A_1}, \ldots, \flat\flat\pi_{A_n} \rangle \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \phi_{\flat\mathbf{A}} \circ \langle \flat\pi_{\flat A_1}, \ldots, \flat\pi_{\flat A_n} \rangle \circ \flat\langle \flat\pi_{A_1}, \ldots, \flat\pi_{A_n} \rangle \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

(Sublemma 20)

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \phi_{\flat\mathbf{A}} \circ \phi^{-1}_{\flat\mathbf{A}} \circ \flat(\phi^{-1}_{\mathbf{A}}) \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \flat(\phi^{-1}_{\mathbf{A}}) \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \flat(\phi_{\mathbf{A}}) \circ \flat(\phi_{\mathbf{A}})^{-1} \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \varepsilon_{\flat(A_1 \times \ldots \times A_n)} \circ \delta_{A_1 \times \ldots \times A_n} \circ \phi_{\mathbf{A}}$$

$$= f \circ \phi^{-1}_{\mathbf{A}} \circ \phi_{\mathbf{A}} \quad \text{(Right Counit Law)}$$

$$= f$$

3. $(g^*\langle f_1^*, ..., f_n^*\rangle = (g\langle f_1^*, ..., f_n^*\rangle)^*)$. The proof makes use of the Coassociativity Law. We give only more perspicuous proof for $m = n = 1$. Given $f : \flat A \to B$ and $g : \flat B \to C$, we have:

$$\begin{aligned} g^* \circ f^* &\equiv \flat g \circ \phi \circ \delta_B \circ \flat f \circ \phi \circ \delta_A \\ &= \flat g \circ \delta_B \circ \flat f \circ \delta_A \\ &= \flat g \circ \flat\flat f \circ \delta_{\flat A} \circ \delta_A \\ &= \flat g \circ \flat\flat f \circ \flat\delta_A \circ \delta_A && \text{(Coassociativity)} \\ &= \flat(g \circ \flat f \circ \delta_A) \circ \delta_A \\ &= \flat(g \circ \flat f \circ \phi \circ \delta_A) \circ \phi \circ \delta_A \\ &= \flat(g \circ f^*) \circ \phi \circ \delta_A \\ &= (g \circ f^*)^* \end{aligned}$$

□

3.2 Interpretation

For the remainder of the chapter, let $\mathcal{E}$ be a topos, with subobject classifier $\top_{\mathcal{E}} : 1_{\mathcal{E}} \to \Omega_{\mathcal{E}}$.

Convention 21. *To denote a monomorphism m with codomain B, we may write $m \rightarrowtail B$, suppressing mention of the domain of m.*

Definition 22. *Let $\phi : X \to \Omega_{\mathcal{E}}$. Then we write $\{\phi\}$ to denote $\phi^*(\top_{\mathcal{E}})$.*

Making our first use of Convention 21, Definition 22 may be distilled to the pullback diagram

$$\begin{tikzcd} \{\phi\} \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1_{\mathcal{E}} \arrow[d, tail, "\top_{\mathcal{E}}"] \\ X \arrow[r, "\phi"'] & \Omega_{\mathcal{E}} \end{tikzcd}$$

.

Definition 23. *Given $\phi, \psi : X \to \Omega_{\mathcal{E}}$, let*

$$\phi \leqslant_X \psi \Leftrightarrow \{\phi\} \leqslant_X \{\psi\} \quad ,$$

i.e. $\Leftrightarrow$ there is a commutative diagram

$$\begin{tikzcd} \{\phi\} \arrow[rr] \arrow[dr, tail] & & \{\psi\} \arrow[dl, tail] \\ & X & \end{tikzcd} .$$

We are now equipped to interpret MIL. We extend the usual categorical interpretation of higher-order logic (c.f. Jacobs 1999, Lambek and Scott 1988) to our setting. This interpretation is presented in the 'denotational style', and is thus not relativized to variable assignments or possible worlds. (Indeed, interpretations of **MIL** will not in general involve possible worlds. C.f. Section 4.3.)

Definition 24. *Let $\mathbf{T}$ be a theory of MIL. An **interpretation** of $\mathbf{T}$ consists of:*

- *a topos $\mathcal{E}$,*
- *a finite limit-preserving comonad $\flat : \mathcal{E} \to \mathcal{E}$,*
- *an object E of $\mathcal{E}$,*
- *an interpretation function $[\![-]\!]$ that takes types to objects of $\mathcal{E}$ and terms derivable from **T** to arrows of $\mathcal{E}$, with the following specifications:*

The iterpretation function $[\![-]\!]$ is recursively defined on types by:

- $[\![\mathsf{E}]\!] = E$
- $[\![\mathsf{T}]\!] = \Omega_{\mathcal{E}}$
- $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$
- $[\![\flat A]\!] = \flat[\![A]\!]$

The interpretation function $[\![-]\!]$ *takes each term*

$$\Gamma \mid t : B$$

to an arrow

$$[\![\Gamma \mid t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$$

in $\mathcal{E}$*, where* $[\![\Gamma]\!]$ *abbreviates* $[\![A_1]\!] \times ... \times [\![A_n]\!]$ *if* Γ *abbreviates* $x_1 : A_1, ..., x_n : A_n$ *and the empty product (i.e. the terminal object)* $1_{\mathcal{E}}$ *if* Γ *is empty. We may abbreviate* $[\![\Gamma \mid t : B]\!]$ *to* $[\![t]\!]$ *when the context is clear. This action of* $[\![-]\!]$ *on terms is given recursively by:*

- *The term* $\Gamma, x : A, \Delta \mid x : A$ *is assigned to the obvious projection*

$$[\![\Gamma]\!] \times [\![A]\!] \times [\![\Delta]\!] \xrightarrow{\pi} [\![A]\!] \quad .$$

- *Each constant* $\cdot \mid c : A$ *of* $\mathbf{T_c}$ *is assigned to an arrow* $[\![c]\!] : 1_{\mathcal{E}} \to [\![A]\!]$.
- *If* $\cdot \mid c : A$ *is a constant,* $[\![\Gamma \mid c : A]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{!} 1_{\mathcal{E}} \xrightarrow{[\![\cdot \mid c]\!]} [\![A]\!] \quad .$$

- *If* $\Gamma, x : A \mid t : B$ *is a term, then* $[\![\Gamma \mid \lambda x.t : A \to B]\!]$ *is*

$$\lambda_{[\![A]\!]}[\![t]\!] : [\![\Gamma]\!] \to [\![B]\!]^{[\![A]\!]} \quad ,$$

  where $\lambda_{[\![A]\!]}[\![t]\!]$ *denotes the exponential transpose of*

$$[\![\Gamma]\!] \times [\![A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \quad .$$

- *If* $\Gamma \mid t : B^A$ *and* $\Gamma \mid u : A$ *are terms, then* $[\![tu]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t]\!], [\![u]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{eval_{[\![B]\!]}} [\![B]\!] \quad .$$

- *If* $\Gamma \mid t : T$ *and* $\Gamma \mid u : T$ *are terms, then* $[\![t \wedge u]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t]\!], [\![u]\!] \rangle} \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}} \xrightarrow{\wedge_{\mathcal{E}}} \Omega_{\mathcal{E}} \quad ,$$

  and similarly for $\vee$ *and* $\Rightarrow$.

- *If* $\Gamma \mid t : T$ *is a term, then* $[\![\neg t]\!]$ *is*

$$[\![\Gamma]\!] \xrightarrow{[\![t]\!]} \Omega_{\mathcal{E}} \xrightarrow{\neg_{\mathcal{E}}} \Omega_{\mathcal{E}} \quad .$$

- $[\![\top]\!] = \top_{\mathcal{E}} : 1_{\mathcal{E}} \to \Omega_{\mathcal{E}}$ *and* $[\![\bot]\!] = \bot_{\mathcal{E}} : 1_{\mathcal{E}} \to \Omega_{\mathcal{E}}$.

- *If* $\Gamma, x : A \mid t : T$ *is a term, then* $[\![\Gamma \mid \forall x.t : T]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{\lambda_{[\![A]\!]}[\![t]\!]} \Omega_{\mathcal{E}}^{[\![A]\!]} \xrightarrow{\forall_{[\![A]\!]}} \Omega_{\mathcal{E}} \quad ,$$

and similarly for $\exists$.

- *If* $\Gamma \mid t : A$ *and* $\Gamma \mid u : A$ *are terms, then* $[\![t =_A u]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![t]\!], [\![u]\!] \rangle} [\![A]\!] \times [\![A]\!] \xrightarrow{\delta_{[\![A]\!]}} \Omega_{\mathcal{E}} \quad ,$$

where $\delta_{[\![A]\!]}$ *is defined as the characteristic arrow of* $\Delta_{[\![A]\!]} :\equiv \langle \mathrm{id}_{[\![A]\!]}, \mathrm{id}_{[\![A]\!]} \rangle$.

- *If* $x_1 : \flat A_1, ..., x_n : \flat A_n \mid t(x_1, ..., x_n) : B$ *is a term and* $\Gamma \mid s_1 : \flat A_1, ..., \Gamma \mid s_n : \flat A_n$ *are terms, then* $[\![\,\hat{}\,t[\mathbf{s}]]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{\langle [\![s_1]\!], ..., [\![s_n]\!] \rangle} [\![\flat A_1]\!] \times ... \times [\![\flat A_n]\!] \xrightarrow{[\![t(x_1, ..., x_n)]\!]^*} \flat [\![A]\!] \quad .$$

- *If* $\Gamma \mid t : \flat A$ *is a term, then* $[\![\,\check{}\,t]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{[\![t]\!]} \flat [\![A]\!] \xrightarrow{\varepsilon_{[\![A]\!]}} [\![A]\!] \quad .$$

- *If* $\Gamma \mid \phi \vdash \psi$ *is an entailment in* $\mathbf{T_e}$*, then* $[\![\phi]\!] \leqslant_{[\![\Gamma]\!]} [\![\psi]\!]$*, where* $\leqslant_{[\![\Gamma]\!]}$ *is as indicated in Definition 23.*

Remark 25. *Note that* $[\![-]\!]$ *is defined by induction on derivations. We do not know a priori that it is well defined on terms, for the same term may arise from multiple derivations via the* Intension *rule. For instance, the term*

$$\cdot \mid \,\hat{}\, f[c] : \flat B$$

may equally well be derived via Intension *from*

$$\cdot \mid c : \flat A \text{ and } x : \flat A \mid f(x) : B$$

or

$$\cdot \mid c : \flat A, \cdot \mid d : \flat A',\ \textit{and}\ x : \flat A, y : \flat A' \mid f(x) : B \quad .$$

The interpretations of these derivations are

$$[\![x : \flat A \mid f(x) : B]\!]^* \circ [\![c]\!]$$

and

$$[\![x : \flat A, y : \flat A' \mid f(x) : B]\!]^* \circ \langle [\![c]\!], [\![d]\!] \rangle \quad ,$$

respectively, whose equality one may doubt.

Intension *is plainly the only way to generate multiple derivations for the same term. To demonstrate that* $[\![-]\!]$ *is well-defined on terms, then, it is sufficient to show that such* Intension *derivations of the same term are assigned the same interpretation. This is deferred to Section 3.2.1.*

Interpretation of *Box* Since $\Box\phi[\mathbf{x}]$ is defined in MIL, rather than taken as primitive, it is of course assigned an interpretation by $[\![-]\!]$. This interpretation will be shown to be equal to the composite

$$[\![\Gamma]\!] \xrightarrow{[\![\,\hat{}\,\phi[\mathbf{x}]]\!]} \flat\Omega_{\mathcal{E}} \xrightarrow{\chi_{\flat\top_{\mathcal{E}}}} \Omega_{\mathcal{E}} \quad ,$$

an identity which will aid the later proof of soundness.

The proof will use the next lemma.

Lemma 26. *The following is a pullback square:*

$$\begin{tikzcd}
1_{\mathcal{E}} \arrow[r, tail, "\top_{\mathcal{E}}"] \arrow[d, tail, "\top_{\mathcal{E}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_{\mathcal{E}} \arrow[d, tail, "\Delta_{\Omega_{\mathcal{E}}}"] \\
\Omega_{\mathcal{E}} \arrow[r, "\langle \mathrm{id}, \top_{\mathcal{E}} \circ !_{\Omega_{\mathcal{E}}} \rangle"'] & \Omega_{\mathcal{E}} \times \Omega_{\mathcal{E}}
\end{tikzcd} \quad .$$

Proof. Assume $f, g : X \to \Omega_{\mathcal{E}}$ are such that $\langle \mathrm{id}, \top_{\mathcal{E}} \circ !_{\Omega_{\mathcal{E}}} \rangle \circ f = \Delta_{\Omega_{\mathcal{E}}} \circ g$. Then we have

$$
\begin{aligned}
\langle f, \top_{\mathcal{E}} \circ !_X \rangle &= \langle f, \top_{\mathcal{E}} \circ !_{\Omega_{\mathcal{E}}} \circ f \rangle \\
&= \langle \mathrm{id}, \top_{\mathcal{E}} \circ !_{\Omega_{\mathcal{E}}} \rangle \circ f \\
&= \Delta_{\Omega_{\mathcal{E}}} \circ g && \text{(Assumption)} \\
&= \langle \mathrm{id}, \mathrm{id} \rangle \circ g \\
&= \langle g, g \rangle
\end{aligned}
$$

.

Therefore $!_X$ is such that $\top_{\mathcal{E}} \circ !_X = f$ and $\top_{\mathcal{E}} \circ !_X = g$. By terminality of $1_{\mathcal{E}}$, it is the unique such map. □

Lemma 27. *If* $x_1 : \flat A_1, ..., x_n : \flat A_n \mid \phi(x_1, ..., x_n) :$ *T is a term and* $\Gamma \mid s_1 : \flat A_1, ..., \Gamma \mid s_n : \flat A_n$ *are terms, then* $[\![\Box \phi[\mathbf{s}]]\!]$ *is the composite*

$$[\![\Gamma]\!] \xrightarrow{[\![\,\hat{}\,\phi[\mathbf{s}]]\!]} \flat\Omega_{\mathcal{E}} \xrightarrow{\chi_{\flat\top_{\mathcal{E}}}} \Omega_{\mathcal{E}} \quad .$$

Proof. Unwinding the definition of $[\![\Box \phi[\mathbf{s}]]\!]$ and applying some basic manipulations yields the following:

$$
\begin{aligned}
[\![\Box\phi[\mathbf{s}]]\!] &= [\![\,\hat{}\top =_{\flat\mathsf{T}} \hat{}\phi[\mathbf{s}]]\!] \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle [\![\,\hat{}\top]\!], [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle [\![\top]\!]^*, [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle ([\![\cdot \mid \top]\!] \circ !_\Gamma)^*, [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle (\top_{\mathcal{E}} \circ !_\Gamma)^*, [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \top_{\mathcal{E}}^* \circ !_\Gamma, [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle && (!\text{ commutes with } *) \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \top_{\mathcal{E}}^* \circ !_{\flat\Omega_{\mathcal{E}}} \circ [\![\,\hat{}\phi[\mathbf{s}]]\!], [\![\,\hat{}\phi[\mathbf{s}]]\!] \rangle \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \top_{\mathcal{E}}^* \circ !_{\flat\Omega_{\mathcal{E}}}, \mathrm{id}_{\flat\Omega_{\mathcal{E}}} \rangle \circ [\![\,\hat{}\phi[\mathbf{s}]]\!] \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \flat\top_{\mathcal{E}} \circ \phi.\circ !_{\flat\Omega_{\mathcal{E}}}, \mathrm{id}_{\flat\Omega_{\mathcal{E}}} \rangle \circ [\![\,\hat{}\phi[\mathbf{s}]]\!] \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \flat\top_{\mathcal{E}} \circ \flat !_{\Omega_{\mathcal{E}}}, \mathrm{id}_{\flat\Omega_{\mathcal{E}}} \rangle \circ [\![\,\hat{}\phi[\mathbf{s}]]\!] \\
&= \delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \flat\top_{\mathcal{E}} \circ \flat !_{\Omega_{\mathcal{E}}}, \flat\,\mathrm{id}_{\Omega_{\mathcal{E}}} \rangle \circ [\![\,\hat{}\phi[\mathbf{s}]]\!]
\end{aligned}
$$

.

It thus suffices to show that $\delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \flat\top_{\mathcal{E}} \circ \flat !_{\Omega_{\mathcal{E}}}, \flat\,\mathrm{id}_{\Omega_{\mathcal{E}}} \rangle = \chi_{\flat\top_{\mathcal{E}}}$. This is exhibited by the following diagram:

$$
\begin{tikzcd}
\flat 1_{\mathcal{E}} \arrow[r, "\flat\top_{\mathcal{E}}"] \arrow[d, tail, "\flat\top_{\mathcal{E}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \flat\Omega_{\mathcal{E}} \arrow[r, "!"] \arrow[d, tail, "\flat\Delta_{\Omega_{\mathcal{E}}}"', "(=\Delta_{\flat\Omega_{\mathcal{E}}})"] \arrow[dr, phantom, "\lrcorner", very near start] & 1_{\mathcal{E}} \arrow[d, tail, "\top_{\mathcal{E}}"] \\
\flat\Omega_{\mathcal{E}} \arrow[r, "\langle \flat\top_{\mathcal{E}} \circ \flat !_{\Omega_{\mathcal{E}}}, \flat\,\mathrm{id}_{\Omega_{\mathcal{E}}} \rangle"'] & \flat\Omega_{\mathcal{E}} \times \flat\Omega_{\mathcal{E}} \arrow[r, "\delta_{\flat\Omega_{\mathcal{E}}}"] & \Omega_{\mathcal{E}}
\end{tikzcd}
$$

.

The left square is a pullback by Lemma 26 and the pullback-preservation of $\flat$. The right square is a pullback by the definition of $\delta_{\flat\Omega_{\mathcal{E}}}$. Thus, by the two-pullbacks lemma, the outer rectangle is a pullback, meaning $\delta_{\flat\Omega_{\mathcal{E}}} \circ \langle \flat\top_{\mathcal{E}} \circ \flat !_{\Omega_{\mathcal{E}}}, \flat\,\mathrm{id}_{\Omega_{\mathcal{E}}} \rangle = \chi_{\flat\top_{\mathcal{E}}}$. $\square$

Lemma 27 provides a sense in which the arrow $\chi_{\flat\top_{\mathcal{E}}} : \flat\Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$ interprets the 'intensional operator' $\Box$. This categorical interpretation of $\Box$ is derived from Reyes and Zolfaghari (1991).

3.2.1 Well-Definedness of $[\![-]\!]$

We return to the issue, raised in Remark 25, of showing that $[\![-]\!]$ is well-defined on terms.

Since the rule *Intension* is the only way to generate multiple derivations for the same term, the key step is to show that such *Intension* derivations of the same term are assigned the same interpretation.

Our approach will be based on the idea that, given premises for *Intension* s, t and s', t' such that $^\frown t[\mathrm{s}] = {}^\frown t'[\mathrm{s}']$, there must be a third set of *Intension* premises, s'', t'', such that s, t and s', t' are both obtainable from s'', t'' (up to renaming of variables) by a rewriting process involving appropriate weakenings, contractions, and permutations. Then, provably, the derivations for $^\frown t[\mathrm{s}]$ and $^\frown t''[\mathrm{s}'']$, as well as the derivations for $^\frown t''[\mathrm{s}'']$ and $^\frown t'[\mathrm{s}']$, are assigned the same interpretation.

In order to proceed, we define this rewriting process.

Definition 28. *The set R of **reduction sequences** is defined recursively as follows:*

- *For each $\Gamma \mid \mathbf{s} : \flat\mathbf{A}$ and $\mathbf{x} : \flat\mathbf{A} \mid t : B$ such that s and $\mathbf{x}$ are of equal length,*

$$\langle\langle \mathbf{s}, \mathbf{x}, t\rangle\rangle \in R \quad .$$

- *(Weakening): If $\sigma :\equiv \langle ..., \langle \mathbf{s}, \mathbf{x}, t\rangle\rangle \in R$, $x_\alpha : \flat A_\alpha \notin \mathbf{x} : \flat\mathbf{A}$, and $\Gamma \mid s_\alpha : \flat A_\alpha$ is a term, then*

$$\sigma \bullet \langle\langle s_1, ..., s_i, s_\alpha, s_{i+1}, ...s_n, x_1, ..., x_i, x_\alpha, x_{i+1}, ..., x_n, t\rangle\rangle \in R \quad .$$

- *(Contraction): If $\sigma :\equiv \langle ..., \langle s_1, ..., s_i, ..., s_j, ..., s_n, x_1, ..., x_i, ..., x_j, ..., x_n, t\rangle\rangle \in R$, $s_i = s_j$, and $i \neq j$, then*

$$\sigma \bullet \langle\langle s_1, ..., s_i, ..., s_{j-1}, s_{j+1}, ..., s_n, x_1, ..., x_i, ..., x_{j-1}, x_{j+1}, ..., x_n, t[x_i/x_j]\rangle\rangle \in R \quad .$$

  The infix $\bullet$ denotes the concatenation function.

- *(Permutation): If $\sigma :\equiv \langle ..., \langle s_1, ..., s_i, ..., s_j, ..., s_n, x_1, ..., x_i, ..., x_j, ..., x_n, t\rangle\rangle \in R$ and $i \neq j$, then*

$$\sigma \bullet \langle\langle s_1, ..., s_{i-1}, s_j, s_{i+1}, ..., s_{j-1}, s_i, s_{j+1}, ..., s_n,$$

$$x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_n, t\rangle\rangle \in R \quad .$$

If $\sigma \in R$, $\mathrm{first}(\sigma) = \langle \mathbf{s}, \mathbf{x}, t\rangle$, *and* $\mathrm{last}(\sigma) = \langle \mathbf{s}', \mathbf{x}', t'\rangle$, *then* σ *is a* ***reduction (sequence)*** *from/of* $\langle \Gamma \mid \mathbf{s}' : \flat\mathbf{A}', \mathbf{x}' : \flat\mathbf{A}' \mid t' : B\rangle$ *to* $\langle \Gamma \mid \mathbf{s} : \flat\mathbf{A}, \mathbf{x} : \flat\mathbf{A} \mid t : B\rangle$. *Furthermore,* $\langle \Gamma \mid \mathbf{s} : \flat\mathbf{A}, \mathbf{x} : \flat\mathbf{A} \mid t : B\rangle$ *can be said to be a* ***reduct*** *of* $\langle \Gamma \mid \mathbf{s}' : \flat\mathbf{A}', \mathbf{x}' : \flat\mathbf{A}' \mid t' : B\rangle$.

Without further ado, we are ready to state:

Theorem 29. *Let* D, D' *be derivations of the same term* $\Gamma \mid t : B$ *from theory* $\mathbf{T}$. *Then* $[\![D]\!] = [\![D']\!]$.

This justifies the notation $[\![\Gamma \mid t : B]\!]$.

The proof of Theorem 29 requires us to prove the following statements by simultaneous induction:

Theorem 29. (Continued).

- (Weakening Lemma). *Given* $\Gamma \mid t : B$,

$$[\![\Gamma, x : A, \Delta \mid t : B]\!] = [\![\Gamma, \Delta \mid t : B]\!] \circ \langle \pi_\Gamma, \pi_\Delta\rangle$$

- (Contraction Lemma). *Given* $\mathbf{x} : \mathbf{A} \mid t : B$ *such that* $x_i = x_j$ *with* $i < j$,

$$[\![x_1 : A_1, ..., x_{j-1} : A_{j-1}, x_i : A_i, x_{j+1} : A_{j+1}, ...x_n : A_n \mid t : B]\!] =$$
$$[\![\mathbf{x} : \mathbf{A} \mid t : B]\!] \circ \langle \pi_1, ..., \pi_{j-1}, \pi_i, \pi_{j+1}, ..., \pi_n\rangle$$

- (Permutation Lemma). *Given* $\mathbf{x} : \mathbf{A} \mid t : B$,

$$[\![x_1 : A_1, ..., x_{i-1} : A_{i-1}, x_j : A_j, x_{i+1} : A_{i+1}, ..., x_{j-1} : A_{j-1}, x_i : A_i, x_{j+1} : A_{j+1}, ...x_n : A_n \mid t : B]\!] =$$
$$[\![\mathbf{x} : \mathbf{A} \mid t : B]\!] \circ \langle \pi_1, .., \pi_{i-1}, \pi_j, \pi_{i+1}, ..., \pi_{j-1}, \pi_i, \pi_{j+1}, ..., \pi_n\rangle$$

- (Reduction Lemma). *If* $\langle \Gamma \mid \mathbf{s}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid t' : C\rangle$ *is a reduct of* $\langle \Gamma \mid \mathbf{s} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid t : C\rangle$, *then* $[\![t]\!]^* \circ \langle [\![s_1]\!], ..., [\![s_n]\!]\rangle = [\![t']\!]^* \circ \langle [\![s'_1]\!], ..., [\![s'_n]\!]\rangle$.

The proofs of the Weakening, Contraction, and Permutation Lemmas are routine and not explicitly indicated within the overall proof. The proof of the Reduction Lemma is given last for readibility.

Proof. We proceed by induction on the depth of $\Gamma \mid t : B$.

(Base Case). When $\Gamma \mid t : B$ has depth 1, it is either a variable $\Gamma, x : A, \Delta \mid x : A$ or a constant $\Gamma \mid c : B$. When $\Gamma \mid t : B$ is $\Gamma, x : A, \Delta \mid x : A$, its only derivation is

$$\frac{}{\Gamma, x : A, \Delta \mid x : A}\text{LC1} \quad .$$

For *LC1* is the only rule that outputs a variable, and any other application of *LC1* yields a different term. Any two derivations of $\Gamma \mid t : B$ are thus equal to the one provided and have the same interpretation. When t is a constant, uniqueness of interpretation follows by similar considerations.

(Inductive Case). We address one of the routine subcases (*LC3*), and then proceed to the interesting case (*Intension*).

(LC3). When $\Gamma \mid t : B$ is $\Gamma \mid \lambda x.u : A \to C$, then its derivation must end

$$\frac{\Gamma, x : A \mid u : C}{\Gamma \mid \lambda x.u : A \to C}\text{LC3} \quad .$$

For *LC3* is the only rule that outputs a λ-term, and any other application of *LC3* yields a different term. Thus, given derivations D, D' of $\lambda x.u$, we have

$$[\![D]\!] = \lambda_{[\![A]\!]}[\![u]\!] : [\![A \to C]\!] = [\![D']\!] \quad .$$

Since u has depth less than $\lambda x.u$, this $[\![u]\!]$ is well-defined by inductive hypothesis.

(Intension). When t is $^\frown v[\mathbf{u}]$, it must come from the rule *Intension*, for this is the only rule that outputs a term starting with the $^\frown$ symbol. Assume without loss of generality that D is a derivation ending

$$\frac{\Gamma \mid u_1 : \flat B_1, ..., u_n : \flat B_n \qquad \mathbf{u} : \flat \mathbf{B} \mid v(x_1, ..., x_n) : C}{\Gamma \mid \ \hat{} v[\mathbf{u}] : \flat C} \text{Intension} \quad .$$

Assume D' is a derivation ending

$$\frac{\Gamma \mid u_1' : \flat B_1', ..., u_m' : \flat B_m' \qquad \mathbf{u}' : \flat \mathbf{B}' \mid v'(x_1', ..., x_m') : C}{\Gamma \mid \hat{} v'[\mathbf{u}'] \equiv \hat{} v[\mathbf{u}] : \flat C} \text{Intension} \quad .$$

As intimated at the outset, we seek, up to renaming of variables, a common reduct $\mathbf{u}'', \mathbf{x}'', v''$ of $\mathbf{u}', \mathbf{x}', v'$ and $\mathbf{u}, \mathbf{x}, v$. Then

$$[\![D]\!] = [\![v]\!]^* \circ \langle [\![u_1]\!], ..., [\![u_n]\!] \rangle = [\![v'']\!]^* \circ \langle [\![u_1'']\!], ..., [\![u_l'']\!] \rangle = [\![v']\!]^* \circ \langle [\![u_1']\!], ..., [\![u_l']\!] \rangle = [\![D']\!] \quad ,$$

where the second and third equalities use the inductive hypothesis of the Reduction Lemma. This string of equalities will demonstrate that $[\![D]\!] = [\![D']\!]$, and, indeed, complete the proof of well-definedness. Note that $[\![v]\!]^* \circ \langle [\![u_1]\!], ..., [\![u_n]\!] \rangle$, $[\![v'']\!]^* \circ \langle [\![u_1'']\!], ..., [\![u_l'']\!] \rangle$, and $[\![v']\!]^* \circ \langle [\![u_1']\!], ..., [\![u_l']\!] \rangle$ are well-defined by inductive hypothesis.

Let $x_1'' : \flat B_1'', ..., x_l'' : \flat B_l'' \mid v''(\mathbf{x}'') : C$ be the term obtained from v by replacing each repetition of a variable with a fresh variable, with context $x_1'' : \flat B_1'', ..., x_l'' : \flat B_l''$ the ordering of the variables of v'' by occurrence from left to right. Let $\Gamma \mid u_i'' : \flat B_i''$ be $\Gamma \mid u_j : \flat B_j$ if $x_i'' = x_j$ or if x_i'' replaced x_j. This defines the terms $\mathbf{u}''$, all of lesser depth than t, since $\mathbf{u}$ all have lesser depth than t. Furthermore, v'' has the same depth as v, thus less than t. The other required property of $\langle \mathbf{u}'', \mathbf{x}'', v'' \rangle$ is shown by the following sublemma:

Sublemma 30. $\langle \Gamma \mid \mathbf{u}'' : \flat \mathbf{B}'', \mathbf{x}'' : \flat \mathbf{B}'' \mid v'' : C \rangle$ *is a reduct of both* $\langle \Gamma \mid \mathbf{u}' : \flat \mathbf{B}', \mathbf{x}' : \flat \mathbf{B}' \mid v' : C \rangle$ *and* $\langle \Gamma \mid \mathbf{u} : \flat \mathbf{B}, \mathbf{x} : \flat \mathbf{B} \mid v : C \rangle$*, up to renaming of variables.*

Proof. Deferred to later work. (Note: see insert on following page for the proof. -CZ, 6/2019). □

It only remains to show the Reduction Lemma. Again, this technically occurs as part of our ongoing simultaneous induction on the depth of t.

Sublemma 30: Insert

Colin Zwanziger

June, 2019

The following is an insert for display following page 37 of the masters thesis

Zwanziger, C. (2017). *Montague's Intensional Logic as Comonadic Type Theory*. M.S. thesis. Carnegie Mellon University. Pittsburgh, USA.

restating and providing the omitted proof of the technical Sublemma 30 of that work. Continuity of definitions and conventions are assumed.

Sublemma 30. $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$ *is a reduct of both* $\langle \Gamma \mid \mathbf{u}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid v' : C\rangle$ *and* $\langle \Gamma \mid \mathbf{u} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid v : C\rangle$*, up to renaming of variables.*

Proof. That $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$ is a reduct of $\langle \Gamma \mid \mathbf{u} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid v : C\rangle$ is apparent from the definition of $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$. A reduction sequence from $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$ to $\langle \Gamma \mid \mathbf{u} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid v : C\rangle$ is obtained by first traversing the term v'', generating entries of the reduction sequence via the rule *Contraction* until the term v is recovered, followed by searching the list $\mathbf{u}''$, generating entries of the reduction sequence via the rule *Weakening* until all entries of list $\mathbf{u}$ are present (though not necessarily in the order given by $\mathbf{u}$). Finally, one unscrambles the scrambled version of the list $\mathbf{u}$ just obtained, generating entries of the reduction sequence via the rule *Permutation*, until $\mathbf{u}$ is recovered.

To complete the proof, we show that $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$ is a reduct of $\langle \Gamma \mid \mathbf{u}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid v' : C\rangle$, up to renaming of variables. Let $\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle$ be the result of applying the same procedure to $\langle \Gamma \mid \mathbf{u}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid v' : C\rangle$ that we applied to $\langle \Gamma \mid \mathbf{u} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid v : C\rangle$ to obtain $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$. By the same argument that showed $\langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle$ is a reduct of $\langle \Gamma \mid \mathbf{u} : \flat\mathbf{B}, \mathbf{x} : \flat\mathbf{B} \mid v : C\rangle$, we have that $\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle$ is a reduct of $\langle \Gamma \mid \mathbf{u}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid v' : C\rangle$. It thus suffices to show that

$$\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle \approx \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle \quad ,$$

where $\approx$ denotes equality up to renaming of variables.

This is shown by induction on the structure of v as a term of the auxiliary language $\mathbf{MIL}'$, as follows:

(Base Cases). There are two subcases.

- When $v \equiv c$, a constant (inclusive of $\top$ or $\bot$), we have

$$\frown c \equiv \frown c[[\mathbf{u}]/\mathbf{x}] \equiv \frown c[\mathbf{u}] \equiv \frown v[\mathbf{u}] \equiv \frown v'[\mathbf{u}'] \equiv \frown v'[[\mathbf{u}']/\mathbf{x}'] \quad .$$

Thus, $c \equiv v'[[\mathbf{u}']/\mathbf{x}']$. For this to be the case, none of the variables in $\mathbf{x}'$ can actually appear in v', and futhermore, we must have $v' \equiv c$. Consequently,

$$\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle = \langle \cdot, \cdot \mid c : C\rangle = \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle \quad .$$

- When $v \equiv x_i$, a variable, we have

$$\frown [u_i] \equiv \frown x_i[[\mathbf{u}]/\mathbf{x}] \equiv \frown x_i[\mathbf{u}] \equiv \frown v[\mathbf{u}] \equiv \frown v'[\mathbf{u}'] \equiv \frown v'[[\mathbf{u}']/\mathbf{x}'] \quad .$$

Thus, $[u_i] \equiv v'[[\mathbf{u}']/\mathbf{x}']$. For this to be the case, we must have $v' \equiv x'_j$ for some variable x'_j such that $u'_j = u_i$. (We cannot have $v' \equiv [u_i]$, as $[u_i]$ is not derivable from $\mathbf{T_c}$.) Consequently,

$$\begin{aligned}\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle &= \langle \Gamma \mid u'_j : \flat B'_j, x'_j : \flat B'_j \mid x'_j : \flat B'_j\rangle \\ &\approx \langle \Gamma \mid u_i : \flat B_i, x_i : \flat B_i \mid x_i : \flat B_i\rangle \\ &= \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle \quad .\end{aligned}$$

(Inductive Cases). In the interest of concision, we treat select and representative cases (in rough order of difficulty).

- We cannot have $v \equiv [w]$, as v would in this case not be derivable from $\mathbf{T_c}$.
- When $v \equiv \neg\phi$, a negation, we have

$$\frown \neg\phi[[\mathbf{u}]/\mathbf{x}] \equiv \frown v[[\mathbf{u}]/\mathbf{x}] \equiv \frown v[\mathbf{u}] \equiv \frown v'[\mathbf{u}] \equiv \frown v'[[\mathbf{u}]/\mathbf{x}] \quad .$$

Thus, $\neg\phi[[\mathbf{u}]/\mathbf{x}] \equiv v'[[\mathbf{u}]/\mathbf{x}]$. For this to be the case, we clearly must have $v' \equiv \neg\phi'$ for some ϕ', so $\phi[[\mathbf{u}]/\mathbf{x}] \equiv \phi'[[\mathbf{u}]/\mathbf{x}]$. Therefore $\frown \phi[[\mathbf{u}]/\mathbf{x}] \equiv \frown \phi'[[\mathbf{u}]/\mathbf{x}]$. By inductive hypothesis, then,

$$\langle \Gamma \mid \mathbf{u^{3,\phi}} : \flat\mathbf{B^{3,\phi}}, \mathbf{x^{3,\phi}} : \flat\mathbf{B^{3,\phi}} \mid \phi^3 : C\rangle \approx \langle \Gamma \mid \mathbf{u}''^{,\phi} : \flat\mathbf{B}''^{,\phi}, \mathbf{x}''^{,\phi} : \flat\mathbf{B}''^{,\phi} \mid \phi'' : C\rangle \quad .$$

Furthermore, it is clear from the description of $(\neg\phi)^3$ that

$$\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid (\neg\phi)^3 : C\rangle \equiv \langle \Gamma \mid \mathbf{u^{3,\phi}} : \flat\mathbf{B^{3,\phi}}, \mathbf{x^{3,\phi}} : \flat\mathbf{B^{3,\phi}} \mid \neg(\phi^3) : C\rangle \quad ,$$

So, finally, we have

$$\begin{aligned}\langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid v^3 : C\rangle &\equiv \langle \Gamma \mid \mathbf{u^3} : \flat\mathbf{B^3}, \mathbf{x^3} : \flat\mathbf{B^3} \mid (\neg\phi)^3 : C\rangle \\ &\equiv \langle \Gamma \mid \mathbf{u^{3,\phi}} : \flat\mathbf{B^{3,\phi}}, \mathbf{x^{3,\phi}} : \flat\mathbf{B^{3,\phi}} \mid \neg(\phi^3) : C\rangle \\ &\approx \langle \Gamma \mid \mathbf{u}''^{,\phi} : \flat\mathbf{B}''^{,\phi}, \mathbf{x}''^{,\phi} : \flat\mathbf{B}''^{,\phi} \mid \neg(\phi'') : C\rangle \\ &\equiv \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid (\neg\phi)'' : C\rangle \\ &\equiv \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C\rangle \quad ,\end{aligned}$$

where the central equation uses the identity we established using the inductive hypothesis. The cases for $\frown$ and $\smile$ are identical.

- When $v \equiv \phi \wedge \psi$, a conjunction, we have

$$\hat{}(\phi \wedge \psi)[[\mathbf{u}]/\mathbf{x}] \equiv \hat{}v[[\mathbf{u}]/\mathbf{x}] \equiv \hat{}v[\mathbf{u}] \equiv \hat{}v'[\mathbf{u}] \equiv \hat{}v'[[\mathbf{u}]/\mathbf{x}] \quad .$$

Thus, $(\phi \wedge \psi)[[\mathbf{u}]/\mathbf{x}] \equiv v'[[\mathbf{u}]/\mathbf{x}]$. For this to be the case, we clearly must have $v' \equiv \phi' \wedge \psi'$ for some ϕ', ψ'. Then

$$\begin{aligned}\phi[[\mathbf{u}]/\mathbf{x}] \wedge \psi[[\mathbf{u}]/\mathbf{x}] &\equiv (\phi \wedge \psi)[[\mathbf{u}]/\mathbf{x}] \\ &\equiv v'[[\mathbf{u}]/\mathbf{x}] \\ &\equiv (\phi' \wedge \psi')[[\mathbf{u}]/\mathbf{x}] \\ &\equiv \phi'[[\mathbf{u}]/\mathbf{x}] \wedge \psi'[[\mathbf{u}]/\mathbf{x}] \quad ,\end{aligned}$$

so $\phi[[\mathbf{u}]/\mathbf{x}] \equiv \phi'[[\mathbf{u}]/\mathbf{x}]$ and $\psi[[\mathbf{u}]/\mathbf{x}] \equiv \psi'[[\mathbf{u}]/\mathbf{x}]$. Therefore $\hat{}\phi[[\mathbf{u}]/\mathbf{x}] \equiv \hat{}\phi'[[\mathbf{u}]/\mathbf{x}]$ and $\hat{}\psi[[\mathbf{u}]/\mathbf{x}] \equiv \hat{}\psi'[[\mathbf{u}]/\mathbf{x}]$. By inductive hypothesis, then,

$$\langle \Gamma \mid \mathbf{u}^{\mathbf{3},\phi} : \flat\mathbf{B}^{\mathbf{3},\phi}, \mathbf{x}^{\mathbf{3},\phi} : \flat\mathbf{B}^{\mathbf{3},\phi} \mid \phi^3 : C \rangle \approx \langle \Gamma \mid \mathbf{u}''^{,\phi} : \flat\mathbf{B}''^{,\phi}, \mathbf{x}''^{,\phi} : \flat\mathbf{B}''^{,\phi} \mid \phi'' : C \rangle$$

and

$$\langle \Gamma \mid \mathbf{u}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi}, \mathbf{x}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi} \mid \psi^3 : C \rangle \approx \langle \Gamma \mid \mathbf{u}''^{,\psi} : \flat\mathbf{B}''^{,\psi}, \mathbf{x}''^{,\psi} : \flat\mathbf{B}''^{,\psi} \mid \psi'' : C \rangle \quad .$$

The variables of $\mathbf{x}^{\mathbf{3},\phi}$ and $\mathbf{x}^{\mathbf{3},\psi}$ (respectively $\mathbf{x}''^{,\phi}$ and $\mathbf{x}''^{,\psi}$), may not be distinct, so we choose $\mathbf{y}^{\mathbf{3},\psi}$ (resp. $\mathbf{y}''^{,\psi}$) distinct from $\mathbf{x}^{\mathbf{3},\phi}$ (resp. $\mathbf{x}''^{,\psi}$) such that

$$\langle \Gamma \mid \mathbf{u}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi}, \mathbf{y}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi} \mid \psi^3[\mathbf{y}^{\mathbf{3},\psi}/\mathbf{x}^{\mathbf{3},\psi}] : C \rangle \approx \langle \Gamma \mid \mathbf{u}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi}, \mathbf{x}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi} \mid \psi^3 : C \rangle$$

(resp.

$$\langle \Gamma \mid \mathbf{u}''^{,\psi} : \flat\mathbf{B}''^{,\psi}, \mathbf{y}''^{,\psi} : \flat\mathbf{B}''^{,\psi} \mid \psi''[\mathbf{y}''^{,\psi}/\mathbf{x}''^{,\psi}] : C \rangle \approx \langle \Gamma \mid \mathbf{u}''^{,\psi} : \flat\mathbf{B}''^{,\psi}, \mathbf{x}''^{,\psi} : \flat\mathbf{B}''^{,\psi} \mid \psi'' : C \rangle \quad).$$

Then, obviously,

$$\begin{aligned}&\langle \Gamma \mid \mathbf{u}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi}, \mathbf{y}^{\mathbf{3},\psi} : \flat\mathbf{B}^{\mathbf{3},\psi} \mid \psi^3[\mathbf{y}^{\mathbf{3},\psi}/\mathbf{x}^{\mathbf{3},\psi}] : C \rangle \approx \\ &\langle \Gamma \mid \mathbf{u}''^{,\psi} : \flat\mathbf{B}''^{,\psi}, \mathbf{y}''^{,\psi} : \flat\mathbf{B}''^{,\psi} \mid \psi''[\mathbf{y}''^{,\psi}/\mathbf{x}''^{,\psi}] : C \rangle \quad .\end{aligned}$$

Furthermore, it is clear from the description of $(\phi \wedge \psi)^3$ that

$$\begin{aligned}&\langle \Gamma \mid \mathbf{u}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}}, \mathbf{x}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}} \mid (\phi \wedge \psi)^3 : C \rangle \approx \\ &\langle \Gamma \mid \mathbf{u}^{\mathbf{3},\phi} \bullet \mathbf{u}^{\mathbf{3},\psi} : \flat(\mathbf{B}^{\mathbf{3},\phi} \bullet \mathbf{B}^{\mathbf{3},\psi}), \mathbf{x}^{\mathbf{3},\phi} \bullet \mathbf{y}^{\mathbf{3},\psi} : \flat(\mathbf{B}^{\mathbf{3},\phi} \bullet \mathbf{B}^{\mathbf{3},\psi}) \mid \phi^3 \wedge \psi^3[\mathbf{y}^{\mathbf{3},\psi}/\mathbf{x}^{\mathbf{3},\psi}] : C \rangle \quad ,\end{aligned}$$

and similarly for $(\phi \wedge \psi)''$. So, finally, we have

$$\begin{aligned}\langle \Gamma \mid \mathbf{u}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}}, \mathbf{x}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}} \mid v^3 : C \rangle &\equiv \langle \Gamma \mid \mathbf{u}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}}, \mathbf{x}^{\mathbf{3}} : \flat\mathbf{B}^{\mathbf{3}} \mid (\phi \wedge \psi)^3 : C \rangle \\ &\approx \langle \Gamma \mid \mathbf{u}^{\mathbf{3},\phi} \bullet \mathbf{u}^{\mathbf{3},\psi} : \flat(\mathbf{B}^{\mathbf{3},\phi} \bullet \mathbf{B}^{\mathbf{3},\psi}), \\ &\qquad \mathbf{x}^{\mathbf{3},\phi} \bullet \mathbf{y}^{\mathbf{3},\psi} : \flat(\mathbf{B}^{\mathbf{3},\phi} \bullet \mathbf{B}^{\mathbf{3},\psi}) \mid \phi^3 \wedge \psi^3[\mathbf{y}^{\mathbf{3},\psi}/\mathbf{x}^{\mathbf{3},\psi}] : C \rangle \\ &\approx \langle \Gamma \mid \mathbf{u}''^{,\phi} \bullet \mathbf{u}''^{,\psi} : \flat(\mathbf{B}''^{,\phi} \bullet \mathbf{B}''^{,\psi}), \\ &\qquad \mathbf{x}''^{,\phi} \bullet \mathbf{y}''^{,\psi} : \flat(\mathbf{B}''^{,\phi} \bullet \mathbf{B}''^{,\psi}) \mid \phi'' \wedge \psi''[\mathbf{y}''^{,\psi}/\mathbf{x}''^{,\psi}] : C \rangle \\ &\approx \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid (\phi \wedge \psi)'' : C \rangle \\ &\equiv \langle \Gamma \mid \mathbf{u}'' : \flat\mathbf{B}'', \mathbf{x}'' : \flat\mathbf{B}'' \mid v'' : C \rangle \quad ,\end{aligned}$$

where the central equation uses the identities we established using the inductive hypothesis. The cases for $\vee$ and $\Rightarrow$ are identical, and those for functional application and $=$ essentially similar.

$\square$

(Base Case). Trivial

(Inductive Case). When t has depth $n+1$, we must induct additionally on the structure of R, the set of reduction sequences.

- (Base Case): In the case of $\sigma = \langle\langle \mathbf{s}', \mathbf{x}', t' \rangle (= \langle \mathbf{s}, \mathbf{x}, t \rangle)\rangle$, i.e. when σ is a reduction of $\langle \Gamma \mid \mathbf{s}' : \flat\mathbf{B}', \mathbf{x}' : \flat\mathbf{B}' \mid t' : C\rangle$ to itself, $[\![t]\!]^* \circ \langle [\![s_1]\!], ..., [\![s_n]\!]\rangle = [\![t']\!]^* \circ \langle [\![s'_1]\!], ..., [\![s'_n]\!]\rangle$ holds trivially.
- (Weakening): If σ is obtained by the clause *Weakening* of Definition 28, then we have

  $$\tau = \langle\langle \mathbf{s}', \mathbf{x}', t' \rangle, ..., \langle q_1, ..., q_n, y_1, ..., y_n, r\rangle\rangle$$

  such that

  $$\sigma \equiv \tau \bullet \langle\langle q_1, ..., q_i, q_\alpha, q_{i+1}, ..., q_n, y_1, ..., y_i, y_\alpha, y_{i+1}, ..., y_n, r\rangle (= \langle \mathbf{s}, \mathbf{x}, t\rangle)\rangle \quad .$$

  By inductive hypothesis, $[\![t']\!]^* \circ \langle [\![s'_1]\!], ..., [\![s'_n]\!]\rangle = [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!]\rangle$. It thus suffices to show that $[\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!]\rangle = [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_i]\!], [\![q_\alpha]\!], [\![q_{i+1}]\!], ..., [\![q_n]\!]\rangle$. This is established as follows:

  $$\begin{aligned}
  [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!]\rangle &= [\![r]\!]^* \circ \langle \pi_1, ..., \pi_n\rangle \circ \langle [\![q_1]\!], ..., [\![q_i]\!], [\![q_\alpha]\!], [\![q_{i+1}]\!], ..., [\![q_n]\!]\rangle \\
  &= ([\![r]\!] \circ \langle \pi_1, ..., \pi_n\rangle)^* \circ \langle [\![q_1]\!], ..., [\![q_i]\!], [\![q_\alpha]\!], [\![q_{i+1}]\!], ..., [\![q_n]\!]\rangle \\
  &\qquad \text{(Projections commute with } (-)^*\text{)} \\
  &= [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_i]\!], [\![q_\alpha]\!], [\![q_{i+1}]\!], ..., [\![q_n]\!]\rangle \\
  &\qquad \text{(Weakening Lemma)}
  \end{aligned}$$

- (Contraction): If σ is obtained by the clause *Contraction* of Definition 28, then we have

  $$\tau = \langle\langle \mathbf{s}', \mathbf{x}', t' \rangle, ..., \langle q_1, ..., q_i, ..., q_j, ..., q_n, y_1, ..., y_i, ..., y_j, ..., y_n, r\rangle\rangle$$

  such that

  $$\begin{gathered}\sigma \equiv \tau \bullet \langle\langle q_1, ..., q_i, ..., q_{j-1}, q_{j+1}, ..., q_n, \\ y_1, ..., y_i, ..., y_{j-1}, y_{j+1}, ..., y_n, r[y_i/y_j]\rangle (= \langle \mathbf{s}, \mathbf{x}, t\rangle)\rangle\end{gathered} \quad .$$

By inductive hypothesis, $[\![t']\!]^* \circ \langle [\![s'_1]\!], ..., [\![s'_n]\!] \rangle = [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!] \rangle$. It thus suffices to show that $[\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!] \rangle = [\![r[y_i/y_j]]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_{j-1}]\!], [\![q_{j+1}]\!], ..., [\![q_n]\!] \rangle$. This is established as follows:

$$
\begin{aligned}
[\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!] \rangle &= [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_{j-1}]\!], [\![q_i]\!], [\![q_{j+1}]\!], ..., [\![q_n]\!] \rangle && (q_i = q_j) \\
&= [\![r]\!]^* \circ \langle \pi_1, ..., \pi_{j-1}, \pi_i, \pi_{j+1}, ..., \pi_n \rangle \circ \langle [\![q_1]\!], ..., [\![q_{j-1}]\!], [\![q_{j+1}]\!], ..., [\![q_n]\!] \rangle \\
&= ([\![r]\!] \circ \langle \pi_1, ..., \pi_{j-1}, \pi_i, \pi_{j+1}, ..., \pi_n \rangle)^* \circ \langle [\![q_1]\!], ..., [\![q_{j-1}]\!], [\![q_{j+1}]\!], ..., [\![q_n]\!] \rangle \\
&\qquad \text{(Projections commute with } (-)^*\text{)} \\
&= [\![r[y_i/y_j]]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_{j-1}]\!], [\![q_{j+1}]\!], ..., [\![q_n]\!] \rangle \\
&\qquad \text{(Contraction Lemma)}
\end{aligned}
$$

- (Permutation): If σ is obtained by the clause *Permutation* of Definition 28, then we have

$$\tau = \langle \langle \mathbf{s}', \mathbf{x}', t' \rangle, ..., \langle q_1, ..., q_i, ..., q_j, ..., q_n, y_1, ..., y_i, ..., y_j, ..., y_n, r \rangle \rangle$$

such that

$$
\begin{gathered}
\sigma \equiv \tau \bullet \langle \langle q_1, ..., q_{i-1}, q_j, q_{i+1}, ..., q_{j-1}, q_i, q_{j+1}, ..., q_n, \\
y_1, ..., y_{i-1}, y_j, y_{i+1}, ..., y_{j-1}, y_i, y_{j+1}, ..., y_n, r \rangle (= \langle \mathbf{s}, \mathbf{x}, t \rangle) \rangle
\end{gathered}.
$$

By inductive hypothesis, $[\![t']\!]^* \circ \langle [\![s'_1]\!], ..., [\![s'_n]\!] \rangle = [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!] \rangle$. It thus suffices to show that $[\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!] \rangle = [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_j]\!], ..., [\![q_i]\!], ..., [\![q_n]\!] \rangle$. This is established as follows:

$$
\begin{aligned}
[\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_n]\!]\rangle &= [\![r]\!]^* \circ \langle \pi_1, ..., \pi_j, ..., \pi_i, ..., \pi_n\rangle \circ \langle [\![q_1]\!], ..., [\![q_j]\!], ..., [\![q_i]\!], ..., [\![q_n]\!]\rangle \\
&= ([\![r]\!] \circ \langle \pi_1, ..., \pi_j, ..., \pi_i, ..., \pi_n\rangle)^* \circ \langle [\![q_1]\!], ..., [\![q_j]\!], ..., [\![q_i]\!], ..., [\![q_n]\!]\rangle \\
&\qquad\qquad\qquad\qquad \text{(Projections commute with } (-)^*\text{)} \\
&= [\![r]\!]^* \circ \langle [\![q_1]\!], ..., [\![q_j]\!], ..., [\![q_i]\!], ..., [\![q_n]\!]\rangle \\
&\qquad\qquad\qquad\qquad \text{(Permutation Lemma)}
\end{aligned}
$$

□

The omitted proof of the technical Sublemma 30 is not included in this thesis, but will be provided separately at a later date (Note: see insert following page 37 for the proof. -CZ, 6/2019). The remainder of Chapter 3, including the proof of soundness, assumes of the result stated in Sublemma 30, and, consequently, Theorem 29 (well-definedness of $[\![-]\!]$).

3.3 Soundness

We turn to the proof of soundness of $[\![-]\!]$. The following lemma will be required:

Lemma 31 (Substitution Lemma)**.** *Let* ***T*** *be a theory of MIL and* $\Gamma \mid \mathbf{s} : \mathbf{A}$ *and* $\mathbf{x} : \mathbf{A} \mid t : B$ *derivable from* ***T***. *Furthermore, let* $[\![-]\!]$ *be an interpretation of* $\mathbf{T}$. *Then* $[\![t[\mathbf{s}/\mathbf{x}]]\!] = [\![t]\!]\langle [\![s_1]\!], ..., [\![s_n]\!]\rangle$.

Proof. The proof is a routine induction on the structure of t, except for the case where t is derived from $\mathbf{x} : \mathbf{A} \mid \mathbf{u} : \flat\mathbf{A}', \mathbf{y} : \flat\mathbf{A}' \mid v : B$. This key step proceeds as follows:

$$
\begin{aligned}
[\![t[\mathbf{s}/\mathbf{x}]]\!] &= [\![\,\hat{}\,v([u_1],...,[u_n])[\mathbf{s}/\mathbf{x}]]\!] \\
&= [\![\,\hat{}\,v([u_1][\mathbf{s}/\mathbf{x}],...,[u_n][\mathbf{s}/\mathbf{x}])]\!] && \text{(Associativity of marked term substitution)} \\
&= [\![\,\hat{}\,v([u_1[\mathbf{s}/\mathbf{x}]],...,[u_n[\mathbf{s}/\mathbf{x}]])]\!] && \text{(Definition of marked term substitution)} \\
&= [\![v]\!]^* \circ \langle [\![u_1[\mathbf{s}/\mathbf{x}]]\!],...,[\![u_n[\mathbf{s}/\mathbf{x}]]\!]\rangle && \text{(Definition of } [\![-]\!]) \\
&= [\![v]\!]^* \circ \langle [\![u_1]\!] \circ \langle [\![s_1]\!],...,[\![s_n]\!]\rangle,...,[\![u_n]\!] \circ \langle [\![s_1]\!],...,[\![s_n]\!]\rangle\rangle && \text{(Inductive hypothesis)} \\
&= [\![v]\!]^* \circ \langle [\![u_1]\!],...,[\![u_n]\!]\rangle \circ \langle [\![s_1]\!],...,[\![s_n]\!]\rangle \\
&= [\![\,\hat{}\,v([u_1],...,[u_n])]\!] \circ \langle [\![s_1]\!],...,[\![s_n]\!]\rangle && \text{(Definition of } [\![-]\!]) \\
&= [\![t]\!] \circ \langle [\![s_1]\!],...,[\![s_n]\!]\rangle
\end{aligned}
$$

□

Theorem 32 (Soundness)**.** *Let* $\mathbf{T}$ *be a theory of* $\mathbf{MIL}$ *and* $\Gamma \mid \phi \vdash \psi$ *derivable from* $\mathbf{T}$*. Furthermore, let* $[\![-]\!]$ *be an interpretation of* $\mathbf{T}$*. Then* $[\![\phi]\!] \leqslant_{[\![\Gamma]\!]} [\![\psi]\!]$*.*

Proof. The soundness of the higher-order logic principles is standard (given Lemma 31). The reader may consult, inter al., Jacobs (1999) and Lambek and Scott (1988) for details. We check a selection of these, as well as each each axiom and inference rule for modal logic and modal type theory.

Higher-Order Logic

2. $\left(\dfrac{[\![\phi]\!] \leqslant_{[\![\Gamma]\!]} [\![\psi]\!] \qquad \Gamma \mid \mathbf{t} : \mathbf{A}}{[\![\phi[\mathbf{t}/\mathbf{x}]]\!] \leqslant_{[\![\Gamma]\!]} [\![\psi[\mathbf{t}/\mathbf{x}]]\!]} \right)$.

   Proof of clause.

$$
\begin{aligned}
[\![\phi[\mathbf{t}/\mathbf{x}]]\!] &= [\![\phi]\!]\langle [\![t_1]\!],...[\![t_n]\!]\rangle && \text{(Lemma 31)} \\
&\leqslant_{[\![\Gamma]\!]} [\![\psi]\!]\langle [\![t_1]\!],...[\![t_n]\!]\rangle && \text{(Functoriality of pullback)} \\
&= [\![\psi[\mathbf{t}/\mathbf{x}]]\!] && \text{(Lemma 31)}
\end{aligned}
$$

$\square$

8. $(\llbracket \top \rrbracket \leqslant_\Gamma \llbracket (\lambda x.t)x' =_B t[x'/x] \rrbracket)$.

   The following lemma will be useful in checking a number of succeeding clauses, in addition to the current one:

   Lemma 33. *For terms* $\Gamma \mid s, t : A$*,*

   $$\llbracket \top \rrbracket \leqslant_{\llbracket \Gamma \rrbracket} \llbracket t =_A s \rrbracket$$

   iff

   $$\llbracket s \rrbracket = \llbracket t \rrbracket$$

   Proof. $(\Rightarrow)$ Assume $\llbracket \top \rrbracket \leqslant_{\llbracket \Gamma \rrbracket} \llbracket t =_A s \rrbracket$. Then, by pasting together the diagram

$$\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[dr, "\leqslant"] \arrow[drr, bend left, "\pi_2 \circ \leqslant"] \arrow[ddr, bend right, "\{\llbracket \top \rrbracket\}(=\mathrm{id})"'] & & & \\
& \cdot \arrow[r, "\pi_2"] \arrow[d, "\{\llbracket s =_A t \rrbracket\}"] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket A \rrbracket \arrow[r] \arrow[d, "\Delta"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
& \llbracket \Gamma \rrbracket \arrow[r, "\langle \llbracket s \rrbracket, \llbracket t \rrbracket \rangle"] \arrow[rr, bend right, "\llbracket s =_A t \rrbracket"'] & \llbracket A \rrbracket \times \llbracket A \rrbracket \arrow[r, "\delta_{\llbracket A \rrbracket}"] & \Omega
\end{tikzcd}$$

,

   it is apparent that $\langle \llbracket s \rrbracket, \llbracket t \rrbracket \rangle = \langle \llbracket s \rrbracket, \llbracket t \rrbracket \rangle \circ \mathrm{id} = (\Delta \circ \pi_2 \circ \leqslant) = (\langle \mathrm{id}, \mathrm{id} \rangle \circ \pi_2 \circ \leqslant) = \langle \mathrm{id} \circ \pi_2 \circ \leqslant, \mathrm{id} \circ \pi_2 \circ \leqslant \rangle$. Therefore $\llbracket s \rrbracket = (\mathrm{id} \circ \pi_2 \circ \leqslant) = \llbracket t \rrbracket$.

   $(\Leftarrow)$ Assume $\llbracket s \rrbracket = \llbracket t \rrbracket$. Then $\Delta \circ \llbracket s \rrbracket = \langle \mathrm{id}, \mathrm{id} \rangle \circ \llbracket s \rrbracket = \langle \llbracket s \rrbracket, \llbracket s \rrbracket \rangle = \langle \llbracket s \rrbracket, \llbracket s \rrbracket \rangle \circ \mathrm{id} = \langle \llbracket s \rrbracket, \llbracket s \rrbracket \rangle \circ \{\llbracket \top \rrbracket\} = \langle \llbracket s \rrbracket, \llbracket t \rrbracket \rangle \circ \{\llbracket \top \rrbracket\}$. Consequently, by the pullback property of the lefthand square of

$$
\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[drr, bend left, "\llbracket s \rrbracket"] \arrow[dr, dashed] \arrow[ddr, bend right, "\{\llbracket \top \rrbracket\}(=\mathrm{id})"'] & & & \\
& \cdot \arrow[r, "\pi_2"] \arrow[d, "\{\llbracket s =_A t \rrbracket\}"] \arrow[dr, phantom, "\lrcorner" very near start] & \llbracket A \rrbracket \arrow[r] \arrow[d, "\Delta"] \arrow[dr, phantom, "\lrcorner" very near start] & 1 \arrow[d, "\top"] \\
& \llbracket \Gamma \rrbracket \arrow[r, "\langle \llbracket s \rrbracket, \llbracket t \rrbracket (= \llbracket s \rrbracket) \rangle"] \arrow[rr, bend right, "\llbracket s =_A t \rrbracket"'] & \llbracket A \rrbracket \times \llbracket A \rrbracket \arrow[r, "\delta_{\llbracket A \rrbracket}"] & \Omega
\end{tikzcd}
$$

,

we have $\llbracket \top \rrbracket \leqslant_{\llbracket \Gamma \rrbracket} \llbracket t =_A s \rrbracket$.

□

Proof of clause. By Lemma 33, it suffices to show $\llbracket (\lambda x.t)x' \rrbracket = \llbracket t[x'/x] \rrbracket$. This is established as follows:

$$
\begin{aligned}
\llbracket \Gamma, x' : A \mid (\lambda x.t)x' \rrbracket &= eval(\langle \llbracket \Gamma, x' : A \mid (\lambda x.t) \rrbracket, \llbracket \Gamma, x' : A \mid x' \rrbracket \rangle) \\
&= eval(\langle \llbracket \Gamma \mid (\lambda x.t) \rrbracket \circ \pi_{\llbracket \Gamma \rrbracket}, \pi_{\llbracket A \rrbracket} \rangle) \\
&= eval(\llbracket \Gamma \mid (\lambda x.t) \rrbracket \times \mathrm{id}) \\
&= eval(\lambda_{\llbracket A \rrbracket} \llbracket \Gamma, x : A \mid t \rrbracket \times \mathrm{id}) \\
&= \llbracket \Gamma, x : A \mid t \rrbracket \\
&= \llbracket \Gamma, x' : A \mid t[x'/x] \rrbracket
\end{aligned}
$$

□

S4 Modal Logic The validity of the rules for $\Box$ is known, and was established in similar frameworks as early as Reyes and Zolfaghari (1991). The reader may consult Awodey, Kishida, and Kotzsch (2014) for further exposition.

The following lemma is used to check each the axioms for $\Box$.

Lemma 34. *Given* $\flat\Gamma \mid \phi : T$*, we have* $\{\llbracket\Box\phi\rrbracket\} = \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat(\{\llbracket\phi\rrbracket\}))$.

Proof. The proof is given for single-variable contexts. The proof for general contexts is similar.

$$
\begin{aligned}
\{\llbracket\Box\phi\rrbracket\} &= \{\chi_{\flat\top} \circ \llbracket\widehat{\ }\phi\rrbracket\} && \text{(Lemma 27)}\\
&= \{\chi_{\flat\top} \circ \llbracket\phi\rrbracket^*\}\\
&= \{\chi_{\flat\top} \circ \flat\llbracket\phi\rrbracket \circ \delta_{\llbracket\flat\Gamma\rrbracket}\}\\
&= \delta^*_{\llbracket\flat\Gamma\rrbracket}\{\chi_{\flat\top} \circ \flat\llbracket\phi\rrbracket\} && \text{(Naturality)}\\
&= \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat\llbracket\phi\rrbracket^*(\{\chi_{\flat\top}\})) && \text{(Naturality)}\\
&= \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat\llbracket\phi\rrbracket^*(\flat\top)) && (\{-\}\text{ inverse to }\chi_-)\\
&= \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat(\llbracket\phi\rrbracket^*(\top))) && (\flat\text{ preserves pullbacks})\\
&\equiv \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat(\{\llbracket\phi\rrbracket\}))
\end{aligned}
$$

□

Convention 35. *In the proofs immediately following, given* $\phi : X \to \Omega$*, we write simply* ϕ *to mean* $\{\phi\}$.

1. $\dfrac{\llbracket\phi\rrbracket \leqslant_{\llbracket\flat\Gamma\rrbracket} \llbracket\psi\rrbracket}{\llbracket\Box\phi\rrbracket \leqslant_{\llbracket\flat\Gamma\rrbracket} \llbracket\Box\psi\rrbracket}$

   Proof of clause. Assume $\llbracket\phi\rrbracket \leqslant_{\llbracket\flat\Gamma\rrbracket} \llbracket\psi\rrbracket$. Then $\llbracket\Box\phi\rrbracket = \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat\llbracket\phi\rrbracket) \leqslant_{\llbracket\flat\Gamma\rrbracket} \delta^*_{\llbracket\flat\Gamma\rrbracket}(\flat\llbracket\psi\rrbracket) = \llbracket\Box\psi\rrbracket$. The first and third equalities are by Lemma 34 and the second is by functoriality of $\flat$ and $\delta^*_{\llbracket\flat\Gamma\rrbracket}$. □

2. $\llbracket\Box\phi\rrbracket \leqslant_{\llbracket\flat\Gamma\rrbracket} \llbracket\phi\rrbracket$

Proof of clause.

$$\begin{tikzcd}
{[\![\Box\phi]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rr, bend left] & \flat[\![\phi]\!] \arrow[r, "\varepsilon_{[\![\phi]\!]}"] \arrow[d, tail] & {[\![\phi]\!]} \arrow[d, tail] \\
\flat[\![\Gamma]\!] \arrow[r, "\delta_{[\![\Gamma]\!]}"] \arrow[rr, bend right, "1_{\flat[\![\Gamma]\!]}"'] & \flat\flat[\![\Gamma]\!] \arrow[r, "\varepsilon_{\flat[\![\Gamma]\!]}"] & \flat[\![\Gamma]\!]
\end{tikzcd}$$

The left square of the preceding diagram commutes by Lemma 34, and its right square commutes by naturality of ε. The bottom of the diagram is id by the Left Counit Law. Thus, the outside of the diagram exhibits $[\![\Box\phi]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\phi]\!]$. □

3. $[\![\Box\phi]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\Box\Box\phi]\!]$

Proof of clause.

$$\begin{tikzcd}
{[\![\Box\phi]\!]} \arrow[r, dashed] \arrow[d] \arrow[ddr, tail] & {[\![\Box\Box\phi]\!]} \arrow[rr] \arrow[dd, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & & \flat[\![\Box\phi]\!] \arrow[dl] \arrow[dd, tail] \arrow[dl, phantom, near start, "\llcorner"] \\
\flat[\![\phi]\!] \arrow[rr, "\delta_{[\![\phi]\!]}"] \arrow[dd, tail] & & \flat\flat[\![\phi]\!] \arrow[dd, tail] & \\
& \flat[\![\Gamma]\!] \arrow[rr, "\delta_{[\![\Gamma]\!]}"] \arrow[dl, "\delta_{[\![\Gamma]\!]}"] & & \flat\flat[\![\Gamma]\!] \arrow[dl, "\flat\delta_{[\![\Gamma]\!]}"] \\
\flat\flat[\![\Gamma]\!] \arrow[rr, "\delta_{\flat[\![\Gamma]\!]}"'] & & \flat\flat\flat[\![\Gamma]\!] &
\end{tikzcd}$$

In the preceding diagram, the left and back squares commutes by Lemma 34, the front by naturality of δ, the right by Lemma 34 and functoriality of $\flat$, and the bottom by the Coassociativity Law. The back is a pullback by definition, and the right because $\flat$ preserves pullbacks. Thus, the back and right together form a pullback. By chasing the commutativities around the cube, an arrow exhibiting $[\![\Box\phi]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\Box\Box\phi]\!]$ is obtained from the universal property of the latter pullback. $\square$

4. $[\![\Box\phi \wedge \Box\psi]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\Box(\phi \wedge \psi)]\!]$

   Proof of clause.

$$\begin{aligned}
[\![\Box(\phi \wedge \psi)]\!] &= \delta^*\flat([\![\phi \wedge \psi]\!]) && \text{(Lemma 34)} \\
&= \delta^*\flat([\![\phi]\!] \wedge_{\flat[\![\Gamma]\!]} [\![\psi]\!]) && \\
&= \delta^*(\flat[\![\phi]\!] \wedge_{\flat\flat[\![\Gamma]\!]} \flat[\![\psi]\!]) && \text{($\flat$ preserves pullbacks.)} \\
&= \delta^*\flat[\![\phi]\!] \wedge_{\flat[\![\Gamma]\!]} \delta^*\flat[\![\psi]\!] && \text{(δ^* preserves pullbacks.)} \\
&= [\![\Box\phi]\!] \wedge_{\flat[\![\Gamma]\!]} [\![\Box\phi]\!] && \text{(Lemma 34)} \\
&= [\![\Box\phi \wedge \Box\phi]\!] &&
\end{aligned}$$

   $\square$

5. $[\![\top]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\Box\top]\!]$

   Proof of clause. $[\![\Box\top]\!] = \delta^*_{[\![\flat\Gamma]\!]}(\flat[\![\top]\!]) \equiv \delta^*_{[\![\flat\Gamma]\!]}(\flat\,\mathrm{id}) = \delta^*_{[\![\flat\Gamma]\!]}(\mathrm{id}) = \mathrm{id} \equiv [\![\top]\!]$, where the first equality is by Lemma 34. $\square$

S4 Modal Type Theory

1. $[\![\top]\!] \leqslant_{[\![\Gamma]\!]} [\![\widehat{\ }\,\widecheck{\ }[x] =_{\flat A} x]\!]$, for all variables $x : \flat A$.

   Proof of clause. By Lemma 33, it suffices to show $[\![\widehat{\ }\widecheck{\ }[x]]\!] = [\![x]\!]$. This follows from *KC1*. More precisely,

$$
\begin{aligned}
[\![\widehat{\ }\widecheck{\ }[x]]\!] &= [\![\widecheck{\ } x]\!]^* \circ [\![x]\!] && \text{(Definition of } [\![\widehat{\ }\widecheck{\ }[x]]\!]) \\
&= [\![\widecheck{\ } x]\!]^* \circ \mathrm{id}_{[\![\flat A]\!]} && \text{(Definition of } [\![x]\!]) \\
&= [\![\widecheck{\ } x]\!]^* \\
&= (\varepsilon_{[\![A]\!]} \circ [\![x]\!])^* && \text{(Definition of } [\![\widecheck{\ } x]\!]) \\
&= (\varepsilon_{[\![A]\!]} \circ \mathrm{id}_{\flat[\![A]\!]})^* && \text{(Definition of } [\![x]\!]) \\
&= (\varepsilon_{[\![A]\!]})^* \\
&= \mathrm{id}_{\flat[\![A]\!]} && (\textit{KC1}) \\
&= [\![x]\!] && \text{(Definition of } [\![x]\!])
\end{aligned}
$$

□

2. $[\![\top]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\widecheck{\ }\widehat{\ } t =_A t]\!]$, for all $\flat\Gamma \mid t : A$.

   Proof of clause. By Lemma 33, it suffices to show $[\![\widecheck{\ }\widehat{\ } t]\!] = [\![t]\!]$. This follows from *KC2* in a similar fashion as above. □

3. $[\![\top]\!] \leqslant_{[\![\flat\Gamma]\!]} [\![\widehat{\ }(t[(\widehat{\ }\mathbf{s})/\mathbf{x}]) =_{\flat B} (\widehat{\ } t)[(\widehat{\ }\mathbf{s})/\mathbf{x}]]\!]$, where $\flat\Gamma \mid s_i : A_i, 0 \leqslant i \leqslant n, x_0 : \flat A_0, ..., x_n : \flat A_n \mid t : B$.

   Proof of clause. By Lemma 33, it suffices to show $[\![\widehat{\ }(t[(\widehat{\ }\mathbf{s})/\mathbf{x}])]\!] = [\![(\widehat{\ } t)[(\widehat{\ }\mathbf{s})/\mathbf{x}]]\!]$. This follows from *KC3*, again in a similar fashion. □

This concludes the proof of Theorem 32 (soundness).

□

Chapter 4

Example Interpretations of MIL

The categorical semantics of the previous chapter subsumes as special cases the semantics of Montague (1973) and a menagerie of other examples. The present chapter indicates how to recover as instances of the general categorical semantics:

- the semantics of Montague (1973).
- a more general Kripke semantics in which an abitrary preorder replaces the set of 'possible worlds' of Montague (1973).
- the Boolean-valued semantics of Gallin (1975), itself a generalization of Montague (1973).

Other interesting examples have a more topological character (c.f. the sheaf semantics for modal logic of Awodey and Kishida 2008), though these are not explored here. The examples presented here are first discussed in Awodey, Buchholtz, and Zwanziger (2016, conference abstract).

Section 1 recalls the relation between adjunctions and comonads. The examples are most intuitively thought of as comonads arising from adjunctions, and will be presented as such. Section 2 addresses the Kripke semantics of MIL, including the classical semantics of Montague (1973). Section 3 indicates the Boolean- (and indeed Heyting-)valued semantics of MIL.

4.1 Comonads from Adjunctions

In order to give the comonads below a more intuitive form, we introduce each of them as arising from an adjunction. The standard connection between comonads and adjunctions is reviewed here.

Let $U : \mathbf{C} \to \mathbf{D}$ and $F : \mathbf{D} \to \mathbf{C}$ form an adjunction $F \dashv U$ (as pictured) with unit $\eta : \mathrm{id}_{\mathbf{D}} \to UF$ and counit $\varepsilon : FU \to \mathrm{id}_{\mathbf{C}}$.

$$
\begin{tikzcd}
\mathbf{C} \arrow[d, bend left, "U"{name=U}] \\
\mathbf{D} \arrow[u, bend left, "F"{name=F}]
\arrow[from=F, to=U, phantom, "\dashv"]
\end{tikzcd}
$$

Then the following facts hold:

The adjunction $F \dashv U$ induces a comonad $FU : \mathbf{C} \to \mathbf{C}$ with counit simply the counit of the adjunction, $\varepsilon : FU \to \mathrm{id}_C$, and comultiplication $F\eta U : FU \to FUFU$.[1]. Furthermore, every comonad arises from an adjunction (not necessarily unique).

As a right adjoint, U preserves all small limits. Dually, the left adjoint, F, preserves all small colimits.

The proofs are available in most textbooks on category theory, including Awodey (2006) and Mac Lane (1971).

Note that, since the right adjoint U preserves all small limits, the comonad FU preserves finite limits if the left adjoint F does.

In each of the examples below, we will consider an adjuction between toposes

[1]Here $F\eta U : FU \to FUFU$ is the natural transformation with component $F(\eta_{UA}) : FUA \to FUFUA$ at each $A \in \mathbf{C}$

$$\begin{tikzcd} \mathcal{E} \arrow[d, bend left, "f_*"] \arrow[d, phantom, "\dashv"] \\ \mathcal{F} \arrow[u, bend left, "f^*"] \end{tikzcd}$$

such that the left adjoint f^* preserves finite limits. By the foregoing, the induced comonad $f^* f_* : \mathcal{E} \to \mathcal{E}$ is a finite limit-preserving comonad, appropriate for interpreting **MIL**.[2]

4.2 Kripke Semantics for MIL, Including that of Montague (1973)

We now recover the Kripke semantics of **MIL** as a special case of our categorical semantics.

4.2.1 The Category $\mathbf{Set}^W$

The resemblance between the semantics Montague offers for his logic and the topos $\mathbf{Set}^W$, for a set W of 'possible worlds', was noted in Lambek (1988). It is in fact this category which is used in our Kripke semantics.

We thus review the structure of $\mathbf{Set}^W$ for a given set W, referring the reader to, inter al., Mac Lane and Moerdijk (1992) for full detail.

Throughout, for any category $\mathbf{W}$, we replace $\mathbf{Set}^{\mathbf{W}}$ by $\widehat{\mathbf{W}}$ in subscripts.

An object P of $\mathbf{Set}^W$ is a functor $W \to \mathbf{Set}$. Since W is a set, P is determined only by its action on objects, meaning it is nothing but a W-indexed family of sets $(P(w))_{(w \in W)}$. An arrow $\alpha : P \to Q$ of $\mathbf{Set}^W$ is a natural transformation from P to Q. But since W is a set, the

[2]In topos theory, such adjunctions $f^* \dashv f_*$ of toposes, in which the left adjoint f^* preserves finite limits, are called **geometric morphisms**. The categorical approach to the semantics for predicate S4 modal logic that the current work draws upon (whose general formulation came in Reyes and Zolfaghari (1991)), is formulated in terms of geometric morphisms, rather than comonads. Compare also the geometric morphism semantics of Awodey, Kishida, and Kotzsch (2014).

naturality condition is trivial, meaning that α is nothing but a W-indexed family of functions $(\alpha_w : P(w) \to Q(w))_{(w \in W)}$.

Structures in $\mathbf{Set}^W$ are likewise indexed versions of those in $\mathbf{Set}$. Limits, colimits, and exponentials are all calculated pointwise, meaning, e.g. $(P \times Q)(w) \cong P(w) \times Q(w)$, $(P + Q)(w) \cong P(w) + Q(w)$, $1_{\widehat{W}}(w) \cong 1_{\mathbf{Set}} \cong \{*\}$, where $\{*\}$ is some one-element set, and $Q^P(w) \cong Q(w)^{P(w)}$.

The subobject classifier $\Omega_{\widehat{W}}$ is given by $\Omega_{\widehat{W}}(w) \cong \Omega_{\mathbf{Set}} \cong \{0,1\}$, where $\{0,1\}$ is some two-element set.

$\top_{\widehat{W}}$ is given by $(\top_{\widehat{W}})_w = \top_{\mathbf{Set}}$, which is itself given by $\top_{\mathbf{Set}}(*) = 1$. $\bot_{\widehat{W}}$ is given by $(\bot_{\widehat{W}})_w = \bot_{\mathbf{Set}}$, which is itself given by $\bot_{\mathbf{Set}}(*) = 0$. $(\wedge_{\widehat{W}})_w$, $(\vee_{\widehat{W}})_w$, $(\Rightarrow_{\widehat{W}})_w$, and $(\neg_{\widehat{W}})_w$ are the usual Boolean operations on $\{0,1\}$ which exhibit it as an instance of the two-element Boolean algebra, with 0 the bottom and 1 the top.

Given an object P of $\mathbf{Set}^W$, $\forall_P : \Omega_{\widehat{W}}^P \to \Omega_{\widehat{W}}$ is given by

$$(\forall_P)_w(k) = 1$$

iff

$$k(x) = 1 \text{ for all } x \in P(w) \quad ,$$

$\exists_P : \Omega_{\widehat{W}}^P \to \Omega_{\widehat{W}}$ is given by

$$(\exists_P)_w(k) = 1$$

iff

$$k(x) = 1 \text{ for all } x \in P(w) \quad .$$

The Kronecker delta δ_P is given by

$$(\delta_P)_w(\langle x, y \rangle) = 1$$

iff

$$x = y, \text{ where } x, y \in P(w) \quad .$$

4.2.2 Montague's Semantics

We are finally equipped to recover the semantics of Montague (1973) from a suitable comonad on $\mathbf{Set}^W$. Intuitively, this comonad 'gives access' to intensions $i : 1_{\widehat{W}} \to P$ within $\mathbf{Set}^W$. It is induced by a certain fundamental adjunction, well-known in topos theory (c.f. Mac Lane and Moerdijk 1992). The connection to Montague is first developed in Awodey, Buchholtz, and Zwanziger (2016).

The Montagovian Comonad

Let Γ be the functor $\mathrm{Hom}_{\widehat{W}}(1, -) : \mathbf{Set}^W \to \mathbf{Set}$. Since an arrow $c : 1_{\widehat{W}} \to P$ in $\mathbf{Set}^W$ is a family of functions $c_w : \{*\} \to P(w)$, c_w is interchangeable with $c_w(*)$ via the natural isomorphism $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, A) \cong A$, so c is essentially the family of elements $c_w(*) \in P(w)$, an 'intension' of P. Intuitively, then, Γ takes the object P to its set of intensions.

In the other direction, the diagonal or constant presheaf functor, $\Delta : \mathbf{Set} \to \mathbf{Set}^W$ takes a set A to the functor ΔA given by $\Delta A(w) = A$ for all $w \in W$. So $\Delta\Gamma P(w) = \Gamma P$ is again the set of intensions of P.

Remark 36. *The functor $\Delta\Gamma$ can be said to 'give access' to intensions at the local level (at each w) in the following sense: an arrow $\alpha : \Delta\Gamma P \to Q$ will have as its component at world w a function $\alpha_w : \mathrm{Hom}_{\widehat{W}}(1_{\widehat{W}}, P) \to Q(w)$ whose domain is the set of P-intensions.*

For each $A \in \mathbf{Set}$ and $P \in \mathbf{Set}^W$, there is an isomorphism

$$\phi : \mathrm{Hom}_{\widehat{W}}(\Delta A, P) \cong \mathrm{Hom}_{\mathbf{Set}}(A, \Gamma P) \quad .$$

This isomorphism ϕ takes the natural transformation $f : \Delta A \to P$ to the function $\phi(f) : A \to \Gamma P$ which takes $a \in A$ to the global section $\phi(f)(a) : 1_{\widehat{W}} \to P$ whose component at $w \in W$ is specified by $(\phi(f)(a))_w(*) = f_w(a)$. In the other direction, ϕ^{-1} takes the function $g : A \to \Gamma P$ to the natural transformation $\phi^{-1}(g) : \Delta A \to P$ whose component at $w \in \mathbf{W}$ is given by $\phi^{-1}(g)_w(a) = g(a)_w(*)$. The reader may check that ϕ and ϕ^{-1} are inverse and that the isomorphism is natural in A and P, so Δ is left adjoint to Γ.

Furthemore, Δ preserves finite limits, as limits are pointwise in $\mathbf{Set}^{\mathbf{W}}$. Thus, we have a finite-limit preserving comonad $\Delta\Gamma : \mathbf{Set}^W \to \mathbf{Set}^W$.

Let $E \in \mathbf{Set}^W$. Then $\Delta\Gamma$ and E together determine an interpretation of $\mathbf{MIL}$.

Intension

Given an interpretation $[\![c]\!] : 1_{\widehat{W}} \to [\![A]\!]$ of a constant $c : A$, we interpret ${}^\frown c$ as $[\![c]\!]^*$, namely the composite

$$1_{\widehat{W}} \overset{\phi}{\cong} \Delta\Gamma 1_{\widehat{W}} \xrightarrow{\Delta\Gamma[\![c]\!]} \Delta\Gamma[\![A]\!] \quad .$$

Consequently, $[\![{}^\frown c]\!]$ is given by

$$\begin{aligned}
[\![{}^\frown c]\!]_w(*) &= ([\![c]\!]^*)_w(*) \\
&= (\Delta\Gamma([\![c]\!])\phi)_w(*) \\
&= (\Delta\Gamma[\![c]\!])_w \phi_w(*) \\
&= (\Delta\Gamma[\![c]\!])_w(\mathrm{id}_{1_{\widehat{W}}}) \\
&= \Gamma[\![c]\!](\mathrm{id}_{1_{\widehat{W}}}) \\
&= \mathrm{Hom}(1_{\widehat{W}}, [\![c]\!])(\mathrm{id}_{1_{\widehat{W}}}) \\
&= [\![c]\!]\,\mathrm{id}_{1_{\widehat{W}}} \\
&= [\![c]\!]
\end{aligned}$$

That is, $[\![{}^\frown c]\!] = \Delta(\widetilde{[\![c]\!]})$ is that intension which at every w is $[\![c]\!]$.

As for terms with free variables, given an interpretation $[\![x_1 : \flat A_1, ..., x_n : \flat A_n \mid t : B]\!] : \Delta\Gamma[\![A_1]\!] \times ... \times \Delta\Gamma[\![A_n]\!] \to [\![B]\!]$, we interpret ${}^\frown t$ as $[\![t]\!]^*$, namely the composite

$$\Delta\Gamma[\![A_1]\!] \times ... \times \Delta\Gamma[\![A_n]\!] \xrightarrow{\Delta\eta_{\Gamma[\![A_1]\!]} \times ... \times \Delta\eta_{\Gamma[\![A_n]\!]}} \Delta\Gamma\Delta\Gamma[\![A_1]\!] \times ... \times \Delta\Gamma\Delta\Gamma[\![A_n]\!] \overset{\phi}{\cong} \Delta\Gamma(\Delta\Gamma[\![A_1]\!] \times ... \times \Delta\Gamma[\![A_n]\!]) \xrightarrow{\Delta\Gamma[\![t]\!]} \Delta\Gamma[\![B]\!] \quad .$$

Consequently, $[\![{}^\frown t]\!]$ is given by

$$
\begin{aligned}
([\![{}^\frown t]\!]_w \mathbf{a})_{w'}(*) &= (([\![t]\!]^*)_w \mathbf{a})_{w'}(*) \\
&= ((((\Delta\Gamma[\![t]\!])\phi(\Delta\eta_{\Gamma[\![A_1]\!]} \times ... \times \Delta\eta_{\Gamma[\![A_n]\!]}))_w \mathbf{a})_{w'}(*) \\
&= ((\Delta\Gamma[\![t]\!])_w \phi_w (\Delta\eta_{\Gamma[\![A_1]\!]} \times ... \times \Delta\eta_{\Gamma[\![A_n]\!]})_w \mathbf{a})_{w'}(*) \\
&= (\Gamma([\![t]\!])\phi_w (\Delta\eta_{\Gamma[\![A_1]\!]} \times ... \times \Delta\eta_{\Gamma[\![A_n]\!]})_w \mathbf{a})_{w'}(*) \\
&= (\Gamma([\![t]\!])\phi_w \psi_w^{-1}((\Delta\eta_{\Gamma[\![A_1]\!]})_w \times ... \times (\Delta\eta_{\Gamma[\![A_n]\!]}))_w \mathbf{a})_{w'}(*) \\
&= (\Gamma([\![t]\!])\phi_w \psi_w^{-1}((\Delta\eta_{\Gamma[\![A_1]\!]})_w \times ... \times (\Delta\eta_{\Gamma[\![A_n]\!]}))_w \mathbf{a})_{w'}(*) \\
&= (\Gamma([\![t]\!])\phi_w \psi_w^{-1}(\eta_{\Gamma[\![A_1]\!]} \times ... \times \eta_{\Gamma[\![A_n]\!]})\mathbf{a})_{w'}(*) \\
&= (\Gamma[\![t]\!]\phi_w \psi_w^{-1}\langle \eta_{\Gamma[\![A_1]\!]}(a_1), ..., \eta_{\Gamma[\![A_n]\!]}(a_n)\rangle)_{w'}(*) \\
&= (\Gamma[\![t]\!]\phi_w \psi_w^{-1}\langle \lambda w.a_1, ..., \lambda w.a_n\rangle)_{w'}(*) \\
&= (\Gamma[\![t]\!]\phi_w \psi_w^{-1}\theta_w^{-1}\lambda w.\langle a_1, ..., a_n\rangle)_{w'}(*) \\
&= (\Gamma[\![t]\!]\lambda w.\langle a_1, ..., a_n\rangle)_{w'}(*) \\
&= (\lambda w.[\![t]\!]_w \langle a_1, ..., a_n\rangle)_{w'}(*) \\
&= [\![t]\!]_{w'}\langle a_1, ..., a_n\rangle
\end{aligned}
$$

That is, $[\![{}^\frown t]\!]$ is that natural transformation which at w takes $\mathbf{A}$ to the intension whose value at each w' is the result of applying $[\![t]\!]_{w'}$ to a.

Remark 37. *Our semantics of Intension thus achieves the treatment of free variables as 'rigid designators' in 'intensional contexts'. This behavior falls out of the comonadic semantics, shedding some light on Montague's treatment of free variables as rigid designators, which on its face seems ad hoc. The present approach is furthermore 'more modular' than Montague's in that its treatment of free variables is achieved together with a standard semantics for function types validating the $\beta\eta$-rules.*

Extension

Now, for the extension operator: For any $P \in \mathbf{Set}^W$, ε_P is by definition $\widetilde{\mathrm{id}_{\Gamma P}}$, which, using the adjunction, is given by $\widetilde{\mathrm{id}_{\Gamma P}}_w(a) = a_w(*)$. That is, ε_P is that natural transformation which at w takes each intension a of P to its value at w.

Revisiting the Restriction on Contexts

Note that the operation used to interpret *Intension*,

$$(-)^* = \flat(-) \circ \flat(\phi^{-1}) \circ \delta_- \circ \phi = \flat(-) \circ \flat(\phi^{-1}) \circ \Delta\eta_{U-} \circ \phi = \flat(-) \circ \flat(\phi^{-1}) \circ \Delta(\tilde{-}) \circ \phi \quad ,$$

when viewed in the latter form, makes sense not just on natural transformations with domain of form $\Delta\Gamma A$ but more generally on those of form ΔA, known as constant functors. If the interpretation of every type were constant, then the interpretation of all contexts would be also, as Δ preserves finite products. In such case, the restriction on contexts in *Intension* would thus be unnecessary.

In the present example, we can impose that all types have constant interpretation by requiring that E does. In this case, all basic types have constant interpretation, since $[\![T]\!] = \Omega_{\widehat{W}} \cong \Delta(\Omega_{Set})$. If $[\![A]\!]$ and $[\![B]\!]$ are constant, then so is $[\![B^A]\!] = [\![B]\!]^{[\![A]\!]}$, as exponentials are pointwise. Finally, if $[\![A]\!]$ is constant, then so clearly is $[\![\flat A]\!] = \Delta\Gamma[\![A]\!]$.

Montague's semantics uses the same set of entities at every world, which corresponds under the present approach to $[\![E]\!]$ being constant. Thus, he does not need the restriction on contexts.

Although a rather minor point in the present example, the restriction on contexts is needed to exploit the full generality of the setup in Example 3.2.

Tense Modalities

Non-S4 modalities may still be given a non-algebraic semantics in the manner of Montague. We illustrate for the tense modalities of Montague (1973): Let I be a set and $\langle J, < \rangle$ a strict total order. Then the future modality $F : \Delta\Gamma\Omega_{\widehat{I\times J}} \to \Delta\Gamma\Omega_{\widehat{I\times J}}$ is given by $(F_w(a))_{\langle i,j\rangle}(*) = \top$ if there exists j' such that $j < j'$ and $a_{\langle i,j'\rangle}(*) = \top$. The past modality $P : \Delta\Gamma\Omega_{\widehat{I\times J}} \to \Delta\Gamma\Omega_{\widehat{I\times J}}$ is given by $(P_w(a))_{\langle i,j\rangle}(*) = \top$ if there exists j' such that $j' < j$ and $a_{\langle i,j'\rangle}(*) = \top$.

4.2.3 Interpretation with an Accessibility Relation

Montague's semantics for the intension and necessity operators use the fact that a single entity, property, etc. may exist at multiple worlds.

However, one might also make use of other relations than equality to make associations across worlds. This tack is taken by Lewis (1968)'s semantics for modal logic, which uses a 'counterpart' relation among entities. As pointed out in Awodey et al. (2014), any presheaf $P : \mathbf{W} \to Set$ on a category $\mathbf{W}$ can be viewed as providing such a relation. For any arrow $f : w \to w'$ in $\mathbf{W}$, $P(f)$ takes $a \in P(w)$ to its 'f-counterpart' in $P(w')$.

In the following example, this presheaf approach is used to give a semantics for arbitrary S4 modalities. Interestingly, an intension in this interpretation is not 'a value at every world' but rather 'a value for every possible path'.

Let $\mathbf{W}$ be a category. We emphasize that when $\mathbf{W}$ is a preorder, its objects may be thought of as worlds and its arrows as providing an accessibility relation. Furthermore, if $\mathbf{W}$ is a general category, its arrows may be thought of as 'accesses', futures, or possible paths.

The inclusion $i : W \hookrightarrow \mathbf{W}$ of the underlying set W into $\mathbf{W}$ induces a functor $i^* : Set^{\mathbf{W}} \to Set^{W}$ by precomposition. That is, i^* is given on objects by $i^*(P) = Pi$. Note that $i^*(P)(w) = Pi(w) = P(w)$; indeed, i^* is the forgetful functor from presheaves on $\mathbf{W}$ to W-indexed family of sets.

As developed in, inter al. Awodey (2006), i^* is part of a string of adjoints $i_! \dashv i^* \dashv i_*$. The action of i_* on objects can be seen from

$$i_*(P)(w) \cong Hom_{\widehat{\mathbf{W}}}(y(w), i_*P) \cong Hom_{\widehat{W}}(i^*(y(w)), P) \quad ,$$

where the first isomorphism is by the Yoneda lemma and the second is because i_* is a right adjoint. Unwinding the definitions, one sees that an $\alpha \in Hom_{\widehat{W}}(i^*(y(w)), P)$ is nothing but a rule which for every w' and $f : w \to w'$ gives an element $\alpha_{w'}(f) \in P(w')$. That is, an element of $i_*(P)(w)[\cong i^*i_*(P)(w)]$ gives a value at the end of every possible path from w, 'an intension of P at w'. In the case where **W** is a preorder, this amounts to a value at every accessible (and not in general every) world.[3]

Both i_* and i^* are a right adjoints, and thus preserve limits. Thus, we have a finite-limit preserving comonad $i^*i_* : \mathbf{Set}^W \to \mathbf{Set}^W$.

Let $E \in \mathbf{Set}^W$. Then i^*i_* and E together specify an interpretation of **MIL**.

For the sake of concision, we merely state facts about $i^* \dashv i_*$. The reader may consult Awodey et al. (2014) for greater detail. The natural isomorphism $\phi : Hom_{\widehat{W}}(i^*A, B) \cong Hom_{\widehat{\mathbf{W}}}(A, i_*B)$ is given by $(\phi(\alpha))_w(a) = \xi$, where ξ is given by $\xi_{w'}(f) = \alpha_{w'}A(f)(a)$. ϕ^{-1} is given by $(\phi^{-1}(\alpha))_w(a) = (\alpha_w(a))_w(Id_w)$.

For an arrow $c : 1_{\widehat{W}} \to P$, we have $i^*(\widetilde{c})_w(*) = \widetilde{c}_w(*) = \xi$, where this last is given by $\xi(f) = c_{w'}(1_{\widetilde{W}}(f)(*)) = c_{w'}(*)$. Thus, the intension of c at w is just the values of c at worlds accessible from w.

More generally, given $t : i^*A \to P$, $i^*(\widetilde{t})_w(a) = \widetilde{t}_w(a) = \xi$, where this last is given by $\xi_{w'}(f) = t_{w'}A(f)(a)$. That is, $i^*(\widetilde{t})$ is that natural transformation which at w takes a to the intension whose value for any possible path $f : w \to w'$ is the result of applying $t_{w'}$ to a's f-counterpart.

ε_P is given by $(\varepsilon_P)_w(a) = a_w(Id_w)$. Thus, ε_P is that natural transformation which at each w takes the intension a to its value on the trivial path.

[3]Those unhappy with this notion of intension but wishing to interpret general S4 modalities might wish to consider a definition of interpretation for *MIL* which uses $\Delta \dashv \Gamma$ for the intension and extension operators and $i^* \dashv i_*$ for the modal operator.

Revisiting the Restriction on Contexts

In light of Section 4.2.2, we might hope to avoid the restriction on contexts in *Intension* and *Box* by amending our notion of interpretation, requiring that $[\![E]\!]$ be $i^*(P)$ for some given $P \in Set^{\mathbf{W}}$. However, this is not on its own enough. If we have $[\![A]\!] = i^*(P)$ and $[\![B]\!] = i^*(Q)$ for given $P, Q \in Set^{\mathbf{W}}$, we need to find a reasonable presheaf structure on $[\![B^A]\!](= [\![B]\!]^{[\![A]\!]} = i^*(Q)^{i^*(P)})$. This is possible when $\mathbf{W}$ is a groupoid, since in that case, $i^*(Q)^{i^*(P)} \cong i^*(Q^P)$. Note that when $\mathbf{W}$ is codiscrete, it is a groupoid, accounting for the option to jettison the context restriction in the Section 4.2.2 semantics. Since we are interested in modal operators arising when $\mathbf{W}$ is not a groupoid, however, there is no obvious way around the restriction.

4.3 Boolean-Valued Semantics

Recall that in Montague (1973), propositions of **MIL** are interpreted as elements of the powerset $\mathbf{2}^W$ of some set W of possible worlds. The logical constants are in effect interpreted using the natural Boolean algebra operations on $\mathbf{2}^W$.

Gallin (1975) generalizes this setup by replacing $\mathbf{2}^W$ with an arbitrary complete Boolean algebra $\mathbf{B}$. Gallin's generalization is recovered in our approach using the comonad $\Delta\Gamma$ induced by

$$\mathrm{Hom}_{\mathrm{Sh}(\mathbf{B})}(1, -) \equiv \Gamma : \mathrm{Sh}(\mathbf{B}) \leftrightarrows \mathbf{Set} : \Delta, \Delta \dashv \Gamma \quad ,$$

the global sections adjunction for the category of sup-topology sheaves on $\mathbf{B}$ (see, inter al., Mac Lane and Moerdijk 1992).

This setup still makes sense when $\mathbf{B}$ is replaced by an arbitrary complete Heyting algebra $\mathbf{H}$, yielding the adjunction

$$\mathrm{Hom}_{\mathrm{Sh}(\mathbf{H})}(1, -) \equiv \Gamma : \mathrm{Sh}(\mathbf{H}) \leftrightarrows \mathbf{Set} : \Delta, \Delta \dashv \Gamma \quad .$$

The Monatagovian comonad of Section 4.2.2 is recovered when $\mathbf{H} \cong \mathbf{2}^W$.

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