Propositional Attitude Operators in Homotopy Type Theory

Extended Abstract

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Colin Zwanziger

Department of Philosophy
Carnegie Mellon University, Pittsburgh, PA, USA
zwanzig@cmu.edu

1 Précis

Since it interprets propositions by sets of possible worlds, the intensional logic of Montague (*locus classicus* 1973) does not distinguish propositions which are true in the same possible worlds. Because of this, the system does not satisfactorily interpret propositional attitude verbs, a fact which has motivated the development of 'hyperintensional' logics (see Fox and Lappin 2008 for a survey).

Below, I isolate a hyperintensional system, Comonadic Homotopy Type Theory (**CHoTT**), which naturally incorporates the intensional logic of Montague with the usual notions of homotopy type theory (see UFP 2013). This system is a fragment of Shulman (2017). From homotopy type theory, we inherit two notions of equality, $\equiv$ and $=$, which we think of as expressing intensional and extensional equalities, respectively. From Montague, we inherit a syntax for intensional operators, which for us will mean operators which respect intensional but not (necessarily) extensional equality. These are used to interpret propositional attitude operators. When interpreting natural language, the intensional equality is chosen to be sufficiently 'granular' that the usual issues are avoided.

2 Comonadic Homotopy Type Theory

As intimated, our system of study, **CHoTT**, combines a version of Montague's intensional logic with homotopy type theory. Since **CHoTT** is a homotopy type theory, it includes two notions of equality: definitional equality, written $\equiv$, and thought of as intensional equality, and typal equality, written $=$, and thought of as extensional equality. Of course, $\equiv$ is stronger than $=$ in a suitable sense. By integrating Montague, we permit 'intensional' operators which do not respect $=$.

In condensed terms, **CHoTT** is the fragment of Shulman (*op. cit.*) consisting of the usual notions of homotopy type theory, together with the comonadic type operator $\flat$, which we think of as an intension operator performing the role of Montague's $\langle s, - \rangle$.

This system is now delineated.

Following in the tradition of Pfenning and Davies (2001) (and including Shulman *op. cit.*), **CHoTT** has two variable judgements,

$$u :: A$$

and

$$x : A$$

We will say (at variance with prior terminology) that "u is an intensional variable of type A," when $u :: A$ and that "x is an extensional variable of type A," when $x : A$. A term in an intensional variable will not be required to respect extensional equality with respect to that variable (that is, 'in that argument').

The hypothetical judgements of **CHoTT** have the form

$$\Delta \mid \Gamma \vdash t : B$$

and

$$\Delta \mid \Gamma \vdash t \equiv u : B$$

where Δ represents a list of intensional typed variables, and Γ a list of extensional typed variables. This $\Delta \mid \Gamma$ is called the context, and we will have, as usual, that a type appearing in the context may depend only on typed variables to its left. So types in Γ can depend on variables in Δ, but not vice versa.

Due to the two variable judgements, there is a duplication of the rules for context extension and variables, with variants for extensional and intensional variables. These are given in the figure below.

$$\frac{}{\cdot \mid \cdot \ \mathsf{ctx}}\ \text{ctx-Emp.}$$

$$\frac{\Delta \mid \Gamma \vdash B : U}{\Delta \mid \Gamma, x : B\ \mathsf{ctx}}\ \text{ctx-Ext.}^e \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma'\ \mathsf{ctx}}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A}\ \text{Var.}^e$$

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta, u :: B \mid \cdot\ \mathsf{ctx}}\ \text{ctx-Ext.}^i \qquad \frac{\Delta, u :: A, \Delta' \mid \Gamma\ \mathsf{ctx}}{\Delta, u :: A, \Delta' \mid \Gamma \vdash u : A}\ \text{Var.}^i$$

Fig. 1. The Extensional ($-^e$) and Intensional ($-^i$) Context Rules

We will import the usual homotopy type theoretical notions (as found in UFP *op. cit.*), including $\prod$- and $\sum$-types (corresponding to the quantifiers $\forall$ and $\exists$), universe polymorphism, $=$-types, higher inductive types (HITs), and univalence. However, the typing rules are assumed to manipulate *extensional variables only*. For instance, the formation rule for $\prod$ is

$$\frac{\Delta \mid \Gamma \vdash A : U \qquad \Delta \mid \Gamma, x : A \vdash B : U}{\Delta \mid \Gamma \vdash \prod_{x:A} B : U}$$

in which $x : A$ is required to be extensional.

Of course, the $=$-types provide our notion of extensional equality. It is the crucial restriction of the $=$-rules to extensional variables that ensures that only terms in extensional variables provably respect $=$. In particular, extensionally identical terms cannot be distinguished by predicates in extensional variables, but may be distinguished by predicates in intensional variables. That is, it is a consequence of the restriction of the $=$-rules to extensional variables that we have the principle

$$\frac{\Delta \mid \Gamma, x : A \vdash B : U \qquad \Delta \mid \Gamma \vdash s, t : A \qquad \Delta \mid \Gamma \vdash p : s =_A t \qquad \Delta \mid \Gamma \vdash q : B[s/x]}{\Delta \mid \Gamma \vdash \ell(s, t, p, q) : B[t/x]}\ \text{Indiscernibility of Identicals}^e$$

in which the variable x of the predicate B is extensional, but *not* the principle

$$\frac{\Delta, u :: A \mid \cdot \vdash B : U \qquad \Delta \mid \cdot \vdash s, t : A \qquad \Delta \mid \cdot \vdash p : s =_A t \qquad \Delta \mid \cdot \vdash q : B[s/x]}{\Delta \mid \Gamma \vdash \ell(s, t, p, q) : B[t/x]}\ \text{Indiscernibility of Identicals}^i$$

in which the variable u of the predicate B is intensional.

Finally, we have a comonad $\flat$ corresponding to Montague's $\langle s, - \rangle$. Terms in the intensional variable $u :: A$ will be interchangeable with terms in the extensional variable $x : \flat A$.

We have the following rules:

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta \mid \Gamma \vdash \flat B : U}\ \flat\text{-Form.} \qquad \frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash t^{\flat} : \flat B}\ \flat\text{-Intro. (Montague's intension operator } {}^{\frown}(-)\text{)}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \Delta \mid \Gamma \vdash s : \flat A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^{\flat}/x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^{\flat} := s\ \mathsf{in}\ t) : B[s/x]}\ \flat\text{-Elim.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \Delta \mid \cdot \vdash s : A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^{\flat}/x]}{\Delta \mid \Gamma \vdash \mathsf{let}\ u^{\flat} := s^{\flat}\ \mathsf{in}\ t \equiv t[s/u] : B[s^{\flat}/x]}\ \flat\text{-}\beta\text{-Conversion}$$

Fig. 2. The Rules for $\flat$

Note that the formation and introduction (intension) rules only apply when no extensional variables are present ("in an intensional context"). Thus, in a well-typed term, any $\flat(-)$ or $(-)^{\flat}$ that appear must have been adduced during a phase of the derivation with an intensional context.

Though the elimination rule is subtle, the reader may take heart that we can derive from it an 'extension' operator corresponding to Montague's ${}^{\smile}(-)$, which (again following Shulman *op. cit.*) we call $(-)_{\flat}$. That is, the rule

$$\frac{\Delta \mid \cdot \vdash B : U \qquad \Delta \mid \Gamma \vdash t : \flat B}{\Delta \mid \Gamma \vdash t_{\flat} : B}\ \flat\text{-Elim.-Simple}$$

is derived by

$$\frac{\dfrac{\Delta \mid \cdot \vdash B : U}{\Delta \mid \Gamma, x : \flat B \vdash B : U}\ \text{Weakening} \qquad \Delta \mid \Gamma \vdash t : \flat B \qquad \dfrac{}{\Delta, u :: B \mid \Gamma \vdash u : B}\ \text{Var.}^{i}}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^{\flat} := t\ \mathsf{in}\ u) \equiv t_{\flat} : B}\ \flat\text{-Elim.}$$

Furthermore, we have the conversion $(t^{\flat})_{\flat} \equiv t$, familiar from Montague. That is, the principle

$$\frac{\Delta \mid \cdot \vdash B : U \qquad \Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash (t^{\flat})_{\flat} \equiv t : B}\ \flat\text{-}\beta\text{-Conv.-Simp.}$$

is obtained by the calculation

$$\begin{aligned} (t^{\flat})_{\flat} &\equiv \mathsf{let}\ u^{\flat} := t^{\flat}\ \mathsf{in}\ u && \text{(Def'n. of } (-)_{\flat}\text{)} \\ &\equiv t && (\flat\text{-}\beta\text{-Conv.}) \end{aligned}$$

3 Comparison with Shulman (*op. cit.*)

CHoTT is a fragment of the type theory of Shulman (*op. cit.*), which includes a further modal operator, $\sharp$. However, Shulman's axioms for $\sharp$ are incompatible with our application. In particular, these imply that intensional variables respect not just $\equiv$, but also $=$ (that is, that the undesirable principle Indiscernibility of Identicals^{i} holds). Also, these axioms imply idempotence of $\flat$ (that is, $\flat\flat A \simeq \flat A$), which does not hold in the familiar model of Montague (*op. cit.*).

It is likely that a natural weakening of the $\sharp$ axioms could avoid these consequences, but this lies beyond the scope of the current work.

4 Interpreting Propositional Attitude Operators

We will now see how to interpret propositional attitude operators via the example of a belief operator.

If "Jane believes that The Morning Star is a planet," and, "The Morning Star is the Evening Star," are both true, it does not follow that "Jane believes that The Evening Star is a planet." **CHoTT** gives a natural interpretation to these sentences where this inference indeed fails.

Like in Montague, we assume a type E of entities. Technically, this is a HIT which has among its constructors $j, m, e : E$ interpreting "Jane," "The Morning Star," and "The Evening Star" and $p : m =_E e$ witnessing the truth of $m =_E e$, which is, of course, the interpretation of "The Morning Star is the Evening Star." We further assume a term $B : E \to \flat U \to U$ interpreting "believes" and a predicate $P : E \to U$ interpreting "is a planet."

This allows us to interpret the sentence, "Jane believes that The Morning Star is a planet," compositionally as

$$B(j, P(m)^\flat)$$

and the sentence, "Jane believes that The Evening Star is a planet," compositionally as

$$B(j, P(e)^\flat)$$

Why does the second not follow from the first? In order to derive the inference, it must be that $B(j, P(x)^\flat)$ is a term in an extensional variable, so that it respects the equality $p : m =_E e$. However, $B(j, P(x)^\flat)$ is ill-typed in this case, since the intension operator $(-)^\flat$ only applies in intensional contexts.

Note finally that we have implicitly assumed *de dicto* readings in this section.

5 Future Work

Several issues remain to be addressed.

The model theory of **CHoTT** is deferred to later work, and will involve the homotopy-theoretic models used for homotopy type theory. In such a model theory, there may be many propositions besides the interpretations of the types 1 and 0. However, when a proposition interpreting the type P is true, there will be a 'path' or 'homotopy' in the model connecting the interpretations of P and 1.

In its basic use of distinct intensional and extensional equalities, the present work bears a relation to other work on (hyper)intensional semantics, including the system of Fox and Lappin (2008). The exact relation to Fox and Lappin is of interest. **CHoTT** does enjoy several advantages; for instance, it allows multiple terms of type $a = b$, which can be thought of as distinct pieces of evidence that a has the same extension as b.

Finally, no interpretation has been suggested for *de re* propositional attitude sentences. To do so satisfactorily would likely involve a more elaborate type theory.

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